Abstract

This project is exploring teachers’ knowledge and perceived practice in teaching an important aspect of algebra in the middle years of schooling – functions, relations and joint variation. Its purpose is to investigate ways to support Australian upper primary teachers’ development of content and pedagogical content knowledge to develop their students’ functional thinking. The need to consider ways to achieve this arose in the context of the *Contemporary Teaching and Learning of Mathematics* (CTLM), a research project funded by the Catholic Education Office (2008-2012).

Informed by empirically- and theoretically-grounded research described in the literature, this project adopted a design-based methodology. Sources of data for the project included an initial survey of 105 Year 5 and 6 teachers. A selection of data on patterns and sequences from the survey form the basis for the findings discussed here. A subsequent collective case study of ten teachers and their upper primary classes is underway.

Analysis of the survey responses from 105 teachers revealed that most teachers found the Year 6 content description on sequences in the ‘Patterns and Algebra’ strand of the new Australian Curriculum relatively easy to understand but less than half indicated that they currently taught this content. On a pattern generalisation task, approximately two thirds of teachers demonstrated content knowledge appropriate to upper primary levels of schooling but less than half indicated a reasonable pedagogical content knowledge.

Keywords

Teacher professional learning; functional thinking; content knowledge; pedagogical content knowledge; algebra; mathematics education; middle years
Introduction

Traditionally, algebra has been viewed as a difficult abstract subject (Greenes, Cavanagh, Dacey, Findell, & Small, 2001) relegated to the secondary years of schooling – “late, abrupt, isolated, and superficial” (Kaput, 2008, p. 6). The practice of teaching arithmetic in the early years of schooling and leaving algebra until later can create significant obstacles to further learning in mathematics (Kieran, 2004), particularly those areas, such as Calculus, where the ability to reason algebraically is important. The teaching and learning of algebra is considered “a major policy concern around the world” (Hodgen, Küchemann, & Brown, 2010). Resistance to algebra in the secondary years of schooling might be reduced if arithmetic and algebra were not misconceived as disjoint subjects and if students were able to develop algebraic thinking at primary levels of schooling (Cai & Moyer, 2008).

Teaching algebra has often taken the form of symbol manipulation techniques which can promote a narrow and instrumental understanding of algebra, rather than algebraic thinking which pervades all dimensions of mathematics. Yet how do teachers, who themselves were schooled via narrow procedural approaches to algebra, develop the ability to “teach a more powerful and general mathematics for understanding”? (Blanton & Kaput, 2008, p. 361) The challenge is to “create a body of knowledge that is learnable and useable by teachers” (Stacey & Chick, 2004, p. 18) which includes an understanding of student conceptions and effective teaching strategies. Attention to teacher professional learning is needed, both for beginning and experienced teachers (Lins & Kaput, 2004). The issues of teachers’ content knowledge and their awareness of students’ difficulties in learning algebra are of increasing importance (Saul, 2008). Ball, Thames, and Phelps (2008) similarly refer to aspects of “subject matter knowledge – in addition to pedagogical content knowledge – that need to be uncovered, mapped, organised, and included in mathematics courses for teachers” (p. 398).

The aim of this project is to explore teachers’ knowledge and perceived practice in teaching an important aspect of algebra in the middle years of schooling – functions, relations and joint variation. Its purpose is to investigate ways to support Australian upper primary teachers’ development of content and pedagogical content knowledge to develop their students’ functional thinking. The need to consider ways to achieve this arose in the context of a research and professional learning project called the *Contemporary Teaching and Learning Mathematics Project* (CTLM), conducted by the Mathematics Teaching and Learning Research Centre at the Australian Catholic University, Melbourne involving 82 Catholic primary schools in Victoria (Australia) and funded by the Catholic Education Office (2008-2012). The project’s major aim is to enhance teacher pedagogical content knowledge in mathematics. The study described here is a sub-project of CTLM that focuses on the professional learning of upper primary teachers in a specific domain of mathematics (algebra). It is intended that insights gained as a result of this research will contribute to accessible resources that support teachers’ professional learning in algebra, address effective implementation of the content and proficiency strands of the new Australian Curriculum for algebra, and support students’ continued learning of algebra to prepare them effectively for learning at secondary levels of schooling.

The following section provides details on the context for the project by describing research on students’ development of functional thinking and on teachers’ professional learning in algebra.

Context and background

Algebra is foundational to mathematics and the development of algebraic thinking from the early years of schooling has emerged as a central theme in contemporary mathematics curriculum (Greenes, et al., 2001). There are differing views on what algebra actually is and what defines algebraic thinking (Kaput, 2008; Kieran, 2004) but generalisation is widely accepted as the cornerstone, the building block of mathematical structure (Kruteskii, 1976). It is the ability of a mathematics learner to see the general in the...
patterns in mathematics: Teachers’ knowledge and perceived practice in the upper primary years

particular (Mason, 1996). Generalisation is suggested as the route to “deep, long-term algebra reform” where students learn with understanding and access the power of algebra, rather than memorise symbol manipulation procedures and (as has happened traditionally) learn to hate algebra (Kaput, 1999).

The two core aspects of algebra are described as the expression of generalisations using conventional symbol systems, and the actions on generalisations (Kaput, 2008). Smith (2008) associated these two aspects with different types of thinking – “representational thinking” and “symbolic thinking” respectively (p. 133). These have been embodied in three strands of algebra, one of which is the study of functions, relations and joint variation (Kaput, 1999). It involves a particular kind of generalising: “describing systematic variation of instances across some domain” (p. 13). Smith (2008) defined functional thinking as a type of “representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalisations of that relationship across instances” (p. 143). Many real-world applications are modelled as functions and significant emphasis is placed on functional thinking in mathematics courses in the later years of schooling. Calculus, to which an understanding of functions is foundational, underlies innovation and economic success across many science and engineering domains, and there is the need for expertise in this area of mathematics (e.g., Mullis, Martin, Gonzalez, & Chrostowski, 2004). The following sub-section reviews literature on students’ development of functional thinking.

The development of functional thinking

Researchers assert that patterns “offer a powerful vehicle for understanding the dependant relations among quantities that underlie mathematical functions” (Moss, Beatty, Barkin, & Shillolo, 2008, p. 156). Early patterning activities are seen as necessary precursors to other types of generalisation in algebra (Greenes, et al., 2001). The development of functional thinking is seen as starting with an understanding of linear functions, which extends naturally from counting experiences involving repeated addition (Smith, 2008). The Principles and Standards for School Mathematics stated that “systematic experience with patterns can build up to an understanding of the idea of function” (National Council of Teachers of Mathematics, 2000, p. 37). In the middle years of schooling, students are to:

- “describe, extend, and make generalisations about geometric and numeric patterns”;
- “represent and analyse patterns and functions, using words, tables, and graphs”;
- “represent, analyse, and generalise a variety of patterns with tables, graphs, words, and, when possible, symbolic rules”;
- “relate and compare different forms of representation for a relationship”;
- “represent the idea of a variable as an unknown quantity using a letter or a symbol”;
- “express mathematical relationships using equations”;
- “model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions” (National Council of Teachers of Mathematics, 2000, p. 158, 222).

The Victorian Essential Learning Standards (VELS) refer to upper primary students constructing and using “rules for sequences based on the previous term, recursion (for example, the next term is three times the last term plus two), and by formula (for example, a term is three times its position in the sequence plus two)”. It also explicitly referred to students being able to “identify relationships between variables and describe them with language and words” (Victorian Curriculum and Assessment Authority, 2007).

The new Australian Curriculum: Mathematics specifically included the content strand “Number and Algebra” and the proficiency strand “Reasoning” all the way from Foundation to Year 10. Explicit
reference is made in the sub-strand “Patterns and Algebra” to the expectations of upper primary students being able to describe, continue, and create patterns (Year 5) and sequences (Year 6), and to describe the rule that creates a sequence (Australian Curriculum Assessment and Reporting Authority, 2009).

Stacey (1989) referred to finding the next item in a growing pattern using step-by-step drawing or counting as “near generalisation” and finding the general rule as “far generalisation” (p. 150). Confrey and Smith (1994) described these two ways of approaching functional situations as co-variation and correspondence. Co-variation describes the relationship between successive items in a pattern – also known as recursive generalisation or a local rule (Mason, 1996) – whereas correspondence perceives the relationship between two quantities or variables (the item/term position number in the pattern/sequence and a quantifiable aspect of the item/term itself – also known as explicit generalisation or a direct or closed or relational rule). Figure 1 provides an example of co-variation and correspondence for a growing pattern.

<table>
<thead>
<tr>
<th>Item position number</th>
<th>Item (e.g., number of blocks)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
</tr>
</tbody>
</table>

**Figure 1:** Two approaches to understanding functional relationships in a sequence

*Source: adapted from Smith (2008, p. 147)*

The following sub-section reviews literature on research of teachers’ mathematics knowledge for teaching algebra.

**Research on teachers' mathematics knowledge for teaching algebra**

Researchers continue to investigate the fundamental question, ‘What do mathematics teachers need to know and be able to do to teach effectively?’ Shulman (1986) described a knowledge beyond content (subject matter) knowledge and pedagogical knowledge that includes “the ways of representing and formulating the subject that make it comprehensible to others”, an understanding of the conceptions, preconceptions, and misconceptions of students of different ages, and “knowledge of strategies most likely to be fruitful in reorganising the understanding of learners” (p. 9). This knowledge has been termed “Pedagogical Content Knowledge” (PCK) and has become part of the research lexicon on teaching and teacher education. It includes mathematical knowledge but of a different kind to that used in everyday life by adults and to that used in other mathematically intensive occupations (Ball & Bass, 2000, p. 88). Shulman (1986) also highlights the importance of curricular knowledge, familiarity with the full range of programs, instructional materials, and tools available for teaching particular concepts at different levels. Ball, Hill and Bass (2005) highlighted the importance of teachers’ skills in not only knowing the content of curriculum but on judging how to utilise it to present, emphasise, sequence and instruct.

Although there is literature that considers the knowledge teachers *ought* to have for teaching mathematics
(e.g., Adler, Davis, Kazima, Parker, & Web, 2005; Ball & Bass, 2000; Carpenter, Fenneman, Peterson, Chian, & Loef, 1989; Nathan & Koellner, 2007) there seems to be far less literature specifically on what knowledge teachers actually do have, and even less literature in the specific domain of knowledge for teaching algebra in the middle years of schooling. A few studies have investigated pre-service teachers’ knowledge for teaching algebra. Two large British studies (n = 154 and n = 201 respectively) of pre-service primary teachers researched their content knowledge of mathematics. Items on algebraic generalisation using words and symbolic notation were completed correctly by less than 50% of participants (Goulding, Rowland, & Barber, 2002). Another study of 58 US elementary pre-service teachers researched their content knowledge in understanding algebraic generalisations and linking symbolic equations to visual growing patterns. The most common difficulties were found to be interpreting what the variables actually represented and identifying the pattern (Rule & Hallagan, 2007).

Nathan and Petrosino (2003) researched 48 pre-service secondary teachers’ content knowledge and pedagogical content knowledge of students’ likely difficulties in algebra. They found that pre-service teachers with higher levels of algebra content knowledge were more likely to believe that worded story problems would be more difficult than symbolic equations to solve, the opposite of what was found in empirical research of students’ learning in algebra. This has been termed the symbol precedence view, as compared to the verbal precedence view. They suggested that educators with advanced content knowledge of algebra but who lack pedagogical content knowledge on how novices learn “tend toward views of student development that align more closely with the organisation of the discipline than with the learning processes of students” (p. 906). It seems that content knowledge of algebra is not sufficient in itself for being able to teach it effectively.

A recent large-scale international six-year comparative study called the Teacher Education and Development Study in Mathematics (TEDS–M) investigated the preparation of primary and lower-secondary teachers for teaching mathematics in 17 countries (Australia did not participate) (Tatto et al., 2012). The research investigated the content knowledge and pedagogical content knowledge of pre-service teachers at the end of their teacher education in four sub-domains, one of which was algebra and functions. Senk et al. (2012) reported on results related to pre-service primary teachers who completed 74 content knowledge items (29% algebra) and 32 pedagogical content knowledge items which included multiple-choice and constructed response formats. They found that those participants who performed at a higher level demonstrated “some familiarity with linear expressions and functions” yet “had limited success applying algebra to geometric situations” (p. 8). Unsurprisingly, they found that pre-service teachers who had undertaken courses to become mathematics specialists performed at higher levels on both content knowledge and pedagogical content knowledge items than those preparing to become generalist teachers. A report of primary level released items from the study included two content knowledge items related to functions, relations and joint variation. Internationally, 77% of pre-service teachers were able to predict the number of matchsticks in the 10th figure of a growing pattern, and 54% were able to find the rule for the number of people that could be seated around n tables. One pedagogical content item that required the selection of an equation to match a growing pattern was completed correctly by only 31% of pre-service teachers (Australian Council for Educational Research, 2010).

A few studies researched practising secondary teachers’ knowledge for teaching algebra. Menzel and Clarke (1999) conducted classroom research on secondary teachers’ pedagogical content knowledge of algebra and reported that it was difficult to find examples of such knowledge from the data set. They noted that each of the teachers observed in the research encouraged rote learning of algebraic procedures and were seldom able to identify specific algebraic concepts that students were most likely to struggle to grasp. They were more likely to refer to “lack of readiness to learn abstract concepts, lack of attention, or lack of practice” (p. 371). They speculated that such teaching could be carried out with minimal pedagogical content knowledge and minimal reflection on reasons for students’ difficulties. Hadjidemetriou and Williams (2002) compared 12 teachers’ pedagogical content knowledge of teaching functions using graphical representations with the actual results of their students (n = 425). Teachers were
asked to judge the difficulty of items, propose a learning sequence and diagnose likely errors and misconceptions. They found that teachers’ lack of content knowledge interfered with their judgements and that there was a mismatch between their perceptions of students’ difficulties and the actual difficulties demonstrated by their students.

Hill, Ball and Schilling (2008) described ongoing research efforts to both conceptualise and measure teachers’ knowledge for teaching mathematics. They built on Shulman’s definitions of different types of knowledge and propose a model of different types of teachers’ mathematical knowledge for teaching, presented in Figure 2.

![Domain map for mathematical knowledge for teaching](image)

**Figure 2:** Domain map for mathematical knowledge for teaching

*Source: from Hill, Ball and Schilling (2008, p. 377)*

Hill et al. (2008) focused on a particular type of pedagogical content knowledge they termed “Knowledge of Content and Students”. Using multiple-choice items in the domains of Number and Algebra, they surveyed hundreds of practising elementary teachers. Although their findings supported their conceptualisation of this type of pedagogical content knowledge as distinct from content or pedagogical knowledge, they reported experiencing significant problems in measuring it. They related these difficulties to the multidimensionality of “knowledge of content and students” and to the limitations of multiple-choice items in the survey. They believed that such questions may not distinguish between teachers’ use of test-taking skills, mathematical reasoning, or knowledge of content and students, in answering items correctly. They suggested that open-ended response items may be more appropriate for researching teachers’ pedagogical content knowledge, even though considerably more expensive for large-scale studies. They asserted that such topic-specific empirical research is important to further our understanding of knowledge of mathematics for teaching.

The following section provides details on the methodology for the project, the research questions it seeks to answer, and the data collection and analysis.

**Research design**

This study adopts a design-based methodology where teachers and researcher experience the project as a collective effort and where teacher learning and student learning are two joint goals (Gravemeijer & Van Eerde, 2009). The three key aspects of this methodology are instructional design and planning, ongoing analysis of classroom events, and retrospective analysis (Cobb, 2000). Teachers and researcher inquire
together “into the nature of learning in a complex system” with the intent of producing “useable knowledge” (Baumgartner, et al., 2003, p. 7) – principles and “explanations of innovative practice” (p. 8). Interactions between materials, teachers and learners are enacted through continuous cycles in order to produce meaningful change in contexts of practice (Baumgartner, et al., 2003).

Informed by empirically- and theoretically-grounded research described in the literature, this study is investigating the professional learning of teachers needed to enable their effective development of students’ functional thinking in the classroom setting. It aims to “address both the pragmatic and highly theoretical issues simultaneously” so that there would be “reflexivity between theory and practice” (Cobb, 2000, p. 308). With the perspective that “the classroom is the primary learning environment for teachers”, this project focuses on “teachers’ learning as it occurs in a social context” (Cobb, 2000, p. 312) – through involvement in ongoing cycles of collaborative planning, implementing, evaluating, and revising lessons (Hiebert & Stigler, 2000). Clarke and Peter (1993, p. 173) stated that “teacher classroom experimentation” is a “critical catalyst for teacher professional growth”. The intent is to understand and improve teacher and student learning in this context, rather than to seek context-free generalisations (Van den Akker, Gravemeijer, McKenney & Niveen, 2006).

The project sought to answer the following central research questions:

1) What are teachers’ perceived current practices for developing students’ functional thinking in their mathematics learning programs?
2) What is the nature of teachers’ content knowledge of functions, relations and joint variation at upper primary levels of mathematics?
3) What is the nature of teachers’ pedagogical content knowledge related to functions, relations and joint variation at upper primary levels of mathematics?
4) What are the facilitators of and impediments to teachers’ effective teaching of functions, relations and joint variation in the upper primary years of schooling?

This article focuses on a discussion of findings that address the first three research questions in relation to patterns and sequences.

The striking similarity between the iterative character of design research methodologies and of models for teacher learning (Gravemeijer & van Eerde, 2009) is evident in a cyclic model for teacher professional growth (Clarke & Peter, 1993) which informed the design of this project. It is based on the premise that teacher learning flourishes where teachers work together (Gravemeijer & van Eerde, 2009). This model describes the process of change in a teacher’s professional growth through the mediating processes of reflection and enactment. Their research with secondary mathematics teachers found that the process of professional growth was marked by the adaptation, rather than the simple adoption, of advocated practices, and that this occurred as a cyclic process of refinement.

Data Collection and analysis

A “descriptive and interpretive” approach (O’Toole & Beckett, 2010, p. 43) to data collection and analysis, with triangulation of multiple sources of data, was implemented. Sources of data for the project included an initial survey of 105 Years 5 and 6 teachers on their understanding of the Australian Curriculum: Mathematics, current teaching practices, content knowledge and pedagogical content knowledge of functions, relations and joint variation, level of confidence in teaching this aspect of algebra, and suggested strategies for their professional learning. The questionnaire was trialled with six upper primary teachers (who were not participants in the CTLM project) to refine the structure and wording of questions on the proforma. The final version used in this study is presented in the Appendix. A selection of data from this initial questionnaire form the basis for the findings discussed in this paper.

A subsequent collective case study of ten teachers is currently underway and is investigating the
professional learning of teachers for developing their students’ functional thinking. It involves iterative cycles of observation and discussion during and after lessons, joint student work analysis, collaborative lesson planning, lesson observation/team teaching, and reflection/evaluation. Teaching team discussions and planning meetings are audio-recorded and include data on teachers’ discussion of: students’ mathematical activity; classroom norms and mathematics practices; revision of instructional materials; and planning for the next instructional activity. These activities have also been designed to solicit informative written data on students’ mathematical thinking and interpretation which are analysed and discussed with colleagues participating in the project (Cobb, 2000).

A later individual interview with these teachers shall explore their perceived facilitators of, and impediments to, their professional learning and their effective teaching of algebra (through case study participation, CTLM involvement, and other areas of influence). A final survey of these ten teachers shall seek data on any changes to teachers’ content knowledge, pedagogical content knowledge, practices and perceptions after their participation in the case study for the year.

Data from the initial questionnaires were analysed using content and interpretive analysis, with the aid of Excel spreadsheets for quantitative analysis and NVivo 9 qualitative analysis software to support line-by-line coding of responses, the refinement of coding, and the adaptation of themes (Creswell, 2007). A process of check-scoring teachers’ responses to assess their level of content knowledge of generalisation of a growing pattern was undertaken by two researchers to increase the reliability of results. A cyclic process of scoring a set of 10 teachers’ responses to Question 3a and discussing results to reach consensus was undertaken for just over half of the questionnaires. Audio and textual analysis from the collective case study is being undertaken cyclically throughout the year to enable emerging ideas to re-shape perspectives, improve instrumentation, and allow for additional data gathering (Miles & Huberman, 1994).

Frameworks for assessing content knowledge and pedagogical content knowledge

The questionnaire used in the study described here contained a number of open-ended response items that sought to investigate different types of knowledge specific to teaching functions, relations and joint variation. These can be described in terms of the previously presented model (Figure 2) developed by Hill et al. (2008):

- Specialised Content Knowledge: Generalising a geometric growing pattern, writing a functional relationship between variables using a symbolic equation, identifying co-variation and correspondence in functional relationships;
- Knowledge of Content and Students: Presenting different levels of possible correct student responses, interpreting and responding to student misconceptions;
- Knowledge of Content and Teaching: Describing appropriate learning experiences for upper primary students, identifying appropriate teaching strategies for exploring functional relationships; and
- Knowledge of curriculum: understanding and applying curriculum content to learning experiences for students.

The open-ended response items can also be described in terms of the Mathematics pedagogical content knowledge framework developed by the previously mentioned TEDS-M project (Tatto et al., 2012) and consisting of three sub-domains:

- Mathematical curricular knowledge: Knowing the school mathematics curriculum about functions, relations and joint variation;
- Knowledge of planning for mathematics teaching and learning: Selecting appropriate activities, identifying different approaches for solving mathematical problems; and
Enacting mathematics for teaching and learning: Explaining or representing mathematics concepts of procedures, and diagnosing students’ responses, including misconceptions.

Recent research by Markworth (2010) sought to develop an empirically substantiated instruction theory about students’ development of functional thinking in the context of geometric growing patterns. She used a design-based research methodology with students in the sixth grade who were anticipated to have had little prior experience with functional thinking or geometric growing patterns. Markworth’s (2010) subsequent learning trajectory was adapted to create the learning progression framework presented in Figure 3.

1. Extend a geometric growing pattern by identifying its physical structure, features that change, and features that remain the same (figural reasoning).

2. Identify quantifiable aspects of items that vary in a geometric growing pattern.

3. Articulate the linear functional relationship between quantifiable aspects of a geometric growing pattern by identifying the change between successive items in the sequence (covariation or recursion).

4. Generalise the linear functional relationship between aspects of a geometric growing pattern by:
   4.1 describing the relationship between a quantifiable aspect of an item and its position in the sequence (correspondence);
   4.2 using symbols or letters to represent variables; or
   4.3 representing the generalisation of a linear function in a full, symbolic equation.

5. Apply an understanding of linear functional relationships between variables to further pattern analysis and multiple representations.

**Figure 3: A learning progression framework of the development of functional thinking with geometric growing patterns**

*Source: adapted from Markworth (2010, p. 253)*

This learning progression was used in the analysis of teachers’ written responses in the initial questionnaires to indicate the level of content knowledge demonstrated. Teachers’ responses to questions relating to pedagogical content knowledge were scored using a generic 4-point rubric, presented in Figure 4, which was adapted for each relevant question.

**Figure 4: A rubric for assessing the level of pedagogical content knowledge demonstrated on students’ development of functional thinking**

*Source: adapted from Downton, Knight, Clarke, & Lewis (2006)*
Results and discussion

The survey of 105 teachers explored different aspects of upper primary teachers’ teaching practice, content knowledge and pedagogical content knowledge about functions, relations and joint variation. A selection of their responses, related to patterns and sequences, is discussed in the following three sub-sections on teachers’: understanding and perceived teaching practice of relevant content from the Australian National Curriculum; stated use of functional language in their teaching; and generalisation of a geometric growing pattern.

‘Patterns and Algebra’ content in the Australian Curriculum

Teachers were asked to respond to the wording of the Year 6 content description in the ‘Patterns and Algebra’ sub-strand of the *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority, 2009) using a Likert scale from 1 to 6 (“easy to understand” to “difficult to understand”):

> Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133).

For this cohort of teachers, the median score was 2, indicating that they found the description quite easy to understand. Teachers were also asked if they currently teach this kind of content and to give an example of an activity they might use to teach it. Table 1 presents the percentages of teachers who gave the content description a particular score and who stated that they currently teach this kind of content. For example, of the 105 teachers in total, 25.7% gave a score of 1 (easy) for the content description, and 81.5% of these teachers also stated that they teach this type of content.

<table>
<thead>
<tr>
<th>Score on Likert scale (1 – 6: easy to difficult to understand)</th>
<th>Percentage of teachers who gave score at each level</th>
<th>Percentage of teachers at each level who stated they taught this type of content</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (easy)</td>
<td>25.7%</td>
<td>81.5%</td>
</tr>
<tr>
<td>2</td>
<td>32.4%</td>
<td>73.5%</td>
</tr>
<tr>
<td>3</td>
<td>13.3%</td>
<td>57.1%</td>
</tr>
<tr>
<td>4</td>
<td>9.5%</td>
<td>60.0%</td>
</tr>
<tr>
<td>5</td>
<td>10.5%</td>
<td>36.4%</td>
</tr>
<tr>
<td>6 (difficult)</td>
<td>3.8%</td>
<td>50.0%</td>
</tr>
<tr>
<td>No score given</td>
<td>4.8%</td>
<td>40.0%</td>
</tr>
</tbody>
</table>

Table 1

*Teachers’ scores for their understanding of “Patterns and Algebra” content and stated teaching practice (n = 105)*

It is perhaps surprising that only two thirds of the teachers in total reported that they currently taught this kind of content. Even though they were asked about content from the recently developed Australian Curriculum, it is important to note that the Victorian curriculum (VELS) also contains standards at upper primary levels of school with even more detail about this area of mathematics.

Nearly 80% of teachers provided an example of an activity for teaching this kind of content, but only 48%
of the examples were deemed as relevant to the teaching of patterns and algebra. For example, the following response was judged as relevant: “Create a growing pattern demonstrated visually and in a table and have students continue the pattern and table of values. Extension is students will write a rule to describe the growing pattern.” The response “Doubling/halving ingredients to cater for different numbers of people” was not considered to be relevant.

Use of functional language in teaching

Teachers were asked to indicate from a list of terms, those which they used explicitly in their teaching. Their responses for terms related to patterns and sequences are presented in Table 2.

Table 2
*Teachers’ stated use of terms related to patterns and sequences*

<table>
<thead>
<tr>
<th>Term related to patterns and sequences</th>
<th>Percentage of teachers who stated they used term explicitly in their teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>sequence</td>
<td>92.4%</td>
</tr>
<tr>
<td>rule</td>
<td>78.1%</td>
</tr>
<tr>
<td>unknown amount/quantity</td>
<td>58.1%</td>
</tr>
<tr>
<td>growing pattern</td>
<td>42.9%</td>
</tr>
<tr>
<td>generalising</td>
<td>39.0%</td>
</tr>
<tr>
<td>variable</td>
<td>33.3%</td>
</tr>
<tr>
<td>function</td>
<td>32.4%</td>
</tr>
<tr>
<td>No response / no terms used</td>
<td>2.9%</td>
</tr>
</tbody>
</table>

It is interesting to note that nearly 93% of teachers indicated that they used the term *sequence* in their teaching but less than half used *growing pattern*. It is likely that this reflects the language used in VELS which referred to *number patterns* and to *sequences*, but not to *geometric* or *growing patterns*. The Australian Curriculum refers to *patterns* in the Year 5 content description and to sequences in Year 6, but these are related to “fractions, decimals and whole numbers” in both year levels. The *Principles and Standards for School Mathematics* explicitly referred to both *geometric* and *numeric patterns* (National Council of Teachers of Mathematics, 2000). Does the wording of the curriculum documents used by Victorian schools limit teachers’ understanding of functions, variables and joint variation? Kuchemann (2010, p. 248) stated that “teachers may not be in the habit of looking for mathematical structure, and will thus need experience of thinking in this way.” Such an approach, facilitated effectively by geometric patterns and visual representations of relationships between variables, as compared to only “compiling lengthy tables of values” may involve less “industrious idyll” and more “uncertainty and risk”, more stressful mathematical thinking, yet perhaps more “challenge and excitement, and the pleasure of engaging with mathematical ideas”.

Joint AARE APERA International Conference, Sydney 2012
Generalisation of a geometric growing pattern

Teachers were asked to provide four possible correct student responses to questions from a growing pattern task (presented in the Appendix). These responses were each analysed and assigned a rubric score from the previously described learning progression (Figure 4). Table 3 provides data on the highest rubric score (for up to four possible written responses) and illustrative responses for each particular score specific to the caterpillar task.

Table 3
Teachers’ highest rubric score for caterpillar task and illustrative responses (n = 105)

<table>
<thead>
<tr>
<th>Highest score on learning progression</th>
<th>Description</th>
<th>Percentage of teachers</th>
<th>Illustrative example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Extend pattern by drawing or adding up stickers on caterpillar</td>
<td>5.7%</td>
<td>“1+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+1”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“5+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+5”</td>
</tr>
<tr>
<td>2</td>
<td>Create sequence or table of quantifiable aspects of caterpillar</td>
<td>9.5%</td>
<td>“Caterpillar # 1  2  3  4…”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Stickers  6  10  14  18…”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“Adding up - skip counting 4s.  6, 10, 14,…”</td>
</tr>
<tr>
<td>3</td>
<td>Explain recursive generalisation by referring to change in caterpillar structure (co-variation)</td>
<td>1.9%</td>
<td>“Start with 6 spots and add 4 more for each new caterpillar”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“Keep adding 4 for each caterpillar”</td>
</tr>
<tr>
<td>4.1</td>
<td>Explain explicit generalisation in words or with a calculation (correspondence)</td>
<td>39.0%</td>
<td>“Number of caterpillars multiplied by 4 then add 2 for each end”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“2 blocks on each end have 10 stickers. Take away those 2 blocks from number of blocks then multiply by 4”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“17 x 4 + 2”</td>
</tr>
<tr>
<td>4.2</td>
<td>Represent generalisation using symbols/letters</td>
<td>31.4%</td>
<td>“4 n + 2”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“(5 x 2) + (c - 2) x 4 = answer”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“6 + 4(n - 1)”</td>
</tr>
<tr>
<td>4.3</td>
<td>Represent generalisation with full, symbolic equation</td>
<td>1.9%</td>
<td>“(a x 4) + 2 = b, where a = no. of blocks, b = no. of stickers”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“4x + 2 = y”</td>
</tr>
<tr>
<td>Unscored response</td>
<td>Incorrect or inappropriate student response</td>
<td>7.6%</td>
<td>“(6 x 17) - (2 x 17)”</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“Look for the pattern and develop an algebraic equation”</td>
</tr>
<tr>
<td>No response</td>
<td></td>
<td>2.9%</td>
<td></td>
</tr>
</tbody>
</table>

Just over 10% of teachers developed sequences (caterpillar number, number of stickers), usually written
in a table format, and used these to generalise recursively (co-variation).

It was found that 72% of the teachers were able to generalise the linear functional relationship between the item number in the pattern (the caterpillar number) and the item itself (the number of stickers on the caterpillar) using correspondence (explicit generalisation). A majority of these teachers (and nearly 40% of the total cohort) described their generalisation in words. Another 31% of teachers used a symbol (typically a letter) to represent their generalisation in an expression, for example, $4n + 2$ or $6 + 4(n - 1)$ ($n =$ caterpillar number). Less than 2% of teachers wrote a full symbolic equation using symbols for both variables, for example $s = 4n + 2$ ($s =$ number of stickers, $n =$ caterpillar number).

The teachers’ examples of (up to) four possible correct student responses were also examined to consider the range of different types of responses given, as an indication of their awareness of the different types of generalisation or stages in a learning progression for these concepts. These results are presented in Table 4.

**Table 4**

*Teachers’ number of different types of student responses for caterpillar task (n = 105)*

<table>
<thead>
<tr>
<th>Number of different types of responses (using learning progressions levels)</th>
<th>Percentage of teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>One type</td>
<td>22.9%</td>
</tr>
<tr>
<td>Two types</td>
<td>44.8%</td>
</tr>
<tr>
<td>Three types</td>
<td>21.0%</td>
</tr>
<tr>
<td>Four types</td>
<td>1.0%</td>
</tr>
<tr>
<td>Unscored response</td>
<td>7.6%</td>
</tr>
<tr>
<td>No response</td>
<td>2.9%</td>
</tr>
</tbody>
</table>

Although 67% of teachers provided two or more different levels of responses, only 18% gave both an example of recursive generalisation (co-variation) and an example of explicit generalisation (correspondence).

By far the most common expression of explicit generalisation related to visualising the structure of the caterpillar as 4 stickers per body segment plus 2 stickers for each end. It was interesting to find that five teachers were able to show two different ways to visualise the structure of the caterpillar and used each of these to find the rule, for example, 4 stickers for each block and 2 for the ends ($4n + 2$), and 5 stickers for each end block and 4 stickers for the blocks in between ($10 + 4(n - 2)$), or 6 stickers for each block take away the dots on the inside where the blocks join together ($6n - 2(n - 1)$).

Teachers were asked to choose the most mathematically sophisticated of their (correct) student response examples and to explain the reasons for their choice. Their responses were analysed and assigned a score using the previously presented rubric, as a way of indicating their pedagogical content knowledge related to students’ learning of generalisation. The results and illustrative responses for each level are presented in Table 5.
Table 5

*Teachers’ explanation of most sophisticated mathematical response to the caterpillar task and illustrative responses (n = 105)*

<table>
<thead>
<tr>
<th>Score on PCK rubric</th>
<th>Description</th>
<th>Percentage of teachers</th>
<th>Illustrative example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incorrect choice</td>
<td>Vague, incorrect or irrelevant explanation</td>
<td>6.7%</td>
<td>“Probably the 1st or last example because it involves 2 operations”</td>
</tr>
<tr>
<td>Correct choice but no explanation</td>
<td>Reference to relevant terms or concepts but key ideas missing or not communicated clearly</td>
<td>16.2%</td>
<td>“4th representation is clear, making a pattern more apparent”</td>
</tr>
<tr>
<td>3</td>
<td>Reference to explicit generalisation (correspondence) e.g., to rules, to efficiency, or to variables/unknowns</td>
<td>21.9%</td>
<td>“3rd - all the student needs to do is change the number of blocks and will have an answer every time. It is similar to 4x+2”</td>
</tr>
<tr>
<td>4</td>
<td>Reference to explicit generalisation (correspondence) and to variables/unknowns</td>
<td>5.7%</td>
<td>“4n+2 involves representing an unknown number with a letter/symbol i.e. Algebra. Shows understanding of pattern and calculating any size caterpillar”</td>
</tr>
<tr>
<td>Unscored response</td>
<td>Incorrect previous student example</td>
<td>7.6%</td>
<td></td>
</tr>
<tr>
<td>No response</td>
<td></td>
<td>21.0%</td>
<td></td>
</tr>
</tbody>
</table>

It was found that less than 30% of teachers referred to correspondence and/or to the use of variables or unknowns. It is perhaps telling that just over 20% of teachers did not make a written response to this particular question. Teachers were then given a scenario related to the caterpillar task in which a student describes an incorrect strategy to find the number of stickers on caterpillar #37:

Caterpillar #37 will have 37 times 4 spots for the top, bottom and sides of the caterpillar, which is 148.

Teachers were asked to describe what they as a teacher might say or do in response to the student. Their written responses were categorised according to their recognition of the response being incorrect and to the level of understanding of addressing the students’ misconception, using the previously presented 4-level rubric. These results are presented in Table 6.
### Table 6

*Teachers’ response to student misconception in the caterpillar task and illustrative responses (n = 105)*

<table>
<thead>
<tr>
<th>Score on PCK rubric</th>
<th>Description</th>
<th>Percentage of teachers</th>
<th>Illustrative example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not recognised as incorrect</td>
<td>Interprets student response as correct</td>
<td>1.9%</td>
<td>“That it is correct”</td>
</tr>
<tr>
<td>Unclear if recognised as incorrect</td>
<td>Vague, incorrect or irrelevant explanation</td>
<td>18.1%</td>
<td>“Can you prove it? Tell me how you came to that answer”</td>
</tr>
<tr>
<td></td>
<td>Reference to relevant terms or concepts but key ideas missing or not communicated clearly</td>
<td>7.6%</td>
<td>“How could you prove to me that your formula(answer) is correct? Without drawing 37 blocks? Could you try your calculations for a smaller number of blocks and find out if they work?”</td>
</tr>
<tr>
<td>Recognised as incorrect</td>
<td>Vague, incorrect or irrelevant explanation</td>
<td>0.0%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Reference to relevant terms or concepts but key ideas missing or not communicated clearly</td>
<td>9.5%</td>
<td>“Maybe get them to visualise and remember that the blocks are connected so you don’t always count the sides. Also ask how many sides does a cube have?”</td>
</tr>
<tr>
<td></td>
<td>Reference to the missing end stickers</td>
<td>33.3%</td>
<td>“I think you may have missed the ones at the end”</td>
</tr>
<tr>
<td></td>
<td>Reference to missing end stickers and to appropriate strategy for student e.g., look at structure of caterpillar, look at smaller caterpillar to check</td>
<td>9.5%</td>
<td>“Firstly, praise that they are close, but need to get some blocks to check. So get connector blocks and join 37 or a small group such as 10+ together, count how many sides and then how many spots. Hopefully from the model they will see the 2 extra spots on the ends”</td>
</tr>
<tr>
<td>No response</td>
<td></td>
<td>20.0%</td>
<td></td>
</tr>
</tbody>
</table>

It was found that 43% of teachers described a response to help the student relate the structure of the caterpillar to the missing number of dots (on the ends of the caterpillar). This is surprising, since 72% had been able to generalise the total number of dots for any number caterpillar. It is possible that some teachers may have used an approach to finding the rule for the growing pattern which involves using an algorithm with a numeric sequence, rather than on the visual structure of the caterpillar, for example, the ‘jump’ between consecutive numbers is the number that goes before the ‘n’ and the difference between the total and the ‘n’ term is the constant. The considerably lower percentage of teachers who demonstrated an effective teaching response to a student misconception seems to illustrate the difference between knowing how to ‘do’ the mathematics (content knowledge) and knowing how to ‘teach’ the mathematics (pedagogical content knowledge).
Conclusion

Analysis of the survey responses for 105 upper primary teachers in Victorian Catholic schools highlighted that a considerable proportion may not be teaching algebra content on patterns and sequences described in relevant curriculum documentation. Their responses indicated that they found the wording of the Australian Curriculum easy enough to understand but struggled to provide appropriate examples of learning activities they could use for teaching the content. Although just over two thirds of teachers were able to complete a patterns and sequence generalisation task appropriate for upper primary students, less than a third demonstrated a reasonable knowledge of the range of possible student responses or levels of understanding. Less than a half of teachers provided an appropriate response to a student’s misconception. This highlighted the difference between knowing how to ‘do’ the mathematics (content knowledge) and knowing how to ‘teach’ the mathematics (pedagogical content knowledge).

Several teachers in their written responses referred to their need to improve their own understanding of algebra and to develop their students’ understanding of it. One teacher wrote, “How to do maths, how to teach such maths”. It would be worthwhile, when considering the professional development of teachers, to pay attention to both their content knowledge of functions, relations and joint variation, and to their pedagogical content knowledge. Ongoing research with a collective case study of ten teachers and their classes continues to investigate ways to address teachers’ professional learning needs in an important area of mathematics so they can prepare students effectively for learning at secondary levels of schooling.

References


Australian Council for Educational Research (2010). Released items: Future teacher mathematics content knowledge (MCK) and mathematics pedagogical content knowledge (MPCK) - Primary: TEDS-M International Study Center, Michigan State University, East Lansing, USA.


Patterns and sequences in mathematics: Teachers' knowledge and perceived practice in the upper primary years

Karina J Wilkie


Hadjidemetriou, C., & Williams, J. (2002). Teachers' pedagogical content knowledge: Graphs from a cognitivist to a situated perspective. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th PME International Conference* (Vol. 3, pp. 57-64).


Patterns and sequences in mathematics: Teachers' knowledge and perceived practice in the upper primary years


Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-M). Amsterdam, the Netherlands: International Association for the Evaluation of Educational Achievement.


Appendix: Questionnaire

1. Patterns and Algebra content description in the Australian National Curriculum (ACARA, 2009)

Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133).

a) For me, this phrase is (please circle):

EASY TO UNDERSTAND

DIFFICULT TO UNDERSTAND

1 2 3 4 5 6

b) I currently teach this kind of content in my class (please circle): YES / NO

c) If I was teaching this kind of content, an example of an activity I would use is:
(There is more room for writing on the back of this questionnaire.)

-------------------------------------------------------------

-------------------------------------------------------------

-------------------------------------------------------------

2. Language I use (explicitly) in my teaching (please tick)

<table>
<thead>
<tr>
<th>Terms</th>
<th>Terms</th>
<th>Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>generalising</td>
<td>unknown amount/quantity</td>
<td>function machine</td>
</tr>
<tr>
<td>growing pattern</td>
<td>variable</td>
<td>input/output</td>
</tr>
<tr>
<td>sequence</td>
<td>function</td>
<td>rule</td>
</tr>
</tbody>
</table>

3. Teaching with growing patterns

Here is a task that explores the continuation, description and generalisation of a geometric growing pattern:

One day my little niece saw a clump of wriggling spotted caterpillars on the branch of a tree. Later she made her own collection of caterpillars with linking blocks and stickers. The first caterpillar was made with 1 block and 6 stickers. The second caterpillar was made with 2 blocks and 10 stickers. She continued to add to her collection:

Caterpillar #1
Caterpillar #2
Caterpillar #3

... 

i) How many stickers are needed for the next caterpillar’s spots (Caterpillar #4)?

ii) How many stickers are needed for Caterpillar #7?

iii) How many stickers are needed for Caterpillar #17?

iv) For any caterpillar number you are given, how do you find the total number of stickers needed for its spots?

(Task adapted from Markworth, 2010)
a) Please give two examples of correct responses you expect students might make to parts iii and iv of this caterpillar task (previous page). Please draw these examples as if you have taken a photocopy of the student’s work. \textit{(There is more room for writing on the back of this questionnaire.)}

iii) How many stickers are needed for Caterpillar #17?
One possible correct student response:

Another possible correct student response:

iv) For any caterpillar number you are given, how do you find the total number of stickers needed for its spots?
One possible correct student response:

Another possible correct student response:

b) Which of your four response examples (above – 1st, 2nd, 3rd or 4th) do you believe is the most sophisticated mathematically? Please explain your reasoning.


c) A student in Year 6 explains that Caterpillar #37 will have 37 times 4 spots for the top, bottom and sides of the caterpillar, which is 148. What might you as the teacher do or say in response?


Patterns and sequences in mathematics: Karina J Wilkie

Teachers’ knowledge and perceived practice in the upper primary years

4. Teaching with function machines

a) Have you used function machines in your teaching? If so, please describe the types of activities you used.

---

b) Here are two input/output tables for the same ‘function machine’:

<table>
<thead>
<tr>
<th>TABLE A</th>
<th>TABLE B</th>
</tr>
</thead>
<tbody>
<tr>
<td>INPUT</td>
<td>OUTPUT</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>n</td>
<td>?</td>
</tr>
</tbody>
</table>

Picture created by Matthew Sexton

What would be the mathematical benefits for students in using one table over the other in a lesson on functional relationships?

TABLE A:

---

TABLE B:

---

5. How do you currently feel about implementing the Patterns and Algebra content of the Australian National Curriculum at your year level?

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6. What would be the most helpful form of support for you in implementing the Patterns and Algebra content of the Australian National Curriculum?

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