Pedagogical content knowledge and the use of examples for teaching ratio

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The upper years of primary school are critical in students' mathematical development. One of the most important concepts is proportional reasoning, which builds upon the foundation of multiplicative thinking, and is central to work in fractions and ratio. To build this understanding in the classroom, teachers need appropriate pedagogical content knowledge (PCK) to choose suitable representations and explanations that highlight mathematical concepts in a meaningful way for students. This PCK is particularly evident in the examples that teachers choose (a) to illustrate concepts initially, and (b) for students to work on during consolidation. This paper presents case studies of two Grade 6 teachers, each giving two lessons on ratio. A framework, derived from the literature, is used to examine the PCK held by the teachers. We focus closely on the examples used by the teachers, since their choice of numerical values, the contexts used, the appropriateness of the example for students, and the aspects of the example highlighted during teaching give insight into teachers' PCK. Both teachers made appropriate numerical choices, but one made much stronger connections to other aspects of mathematics, gave more relevant contexts, and selected problems that better matched her students' ability.

Background

Examples and Affordances

Most teachers use examples in some form or other in their teaching, whether in demonstrating a particular way to solve a problem or setting students certain exercises. This is particularly true for mathematics teaching. For this paper we define an example to be "a specific instantiation of a general principle, chosen in order to illustrate or explore that principle" (Chick, 2007). The intention is that the specific instance will enhance understanding of the principle. Thus, for instance, a teacher might highlight 6, 16, and 54 as examples of even numbers because they are each divisible by two. The point that the teacher is making is not about 6, 16, and 54 *per se,* but about the idea of even numbers having two as a factor. Similarly, most of the tasks and exercises that teachers assign to students are given not because of an interest in the particular answer to any one of them but because of the general principles that they illustrate (so, for instance, the numerical value of 367×32 is not an important outcome of doing the long multiplication compared to learning the procedure).

An overview of the history and role of examples has been given by Bills, Mason, Watson, and Zaslavsky (2006). Of concern to us are the issues of how to choose appropriate examples that allow the general principle to be noticed above the "noise" of the specific details (see Skemp, 1971, pp. 29-30). Rowland and Zaslavsky illustrate this with an example about regrouping in two-digit subtraction, highlighting that certain variations in the digits of the problem 62-38 allow the example to demonstrate the desired feature, but other choices prevent the example being of use for regrouping (cited in Bills et al., 2006). Watson and Mason use the term *dimensions of possible variation* (adapted from Marton, cited in Watson & Mason, 2005), to refer to ways in which an example can change without changing what it exemplifies. Ball (2000) has highlighted that a critical issue for teaching is to decide how to modify a task to make it easier or simpler, while still illustrating the general principle.

Bills et al. (2006, p.138) write that there has been only limited research on teachers' choice of examples. Recent examples of such research include Zazkis and Chernoff (2006), who described a one-on-one teacher-learner situation in which the teacher (also researcher) had to make impromptu but strategic decisions about what examples could be used to help the student understand prime numbers. Similarly, Zaslavsky, Harel, and Manaster (2006) examined how the mathematical knowledge of the teacher influenced the kinds of examples she could use for introducing Pythagoras' Theorem. Chick (2007) looked at the way that a number of teachers used examples in teaching fractions, probability and measurement.

Central to the question of example choice is the issue of awareness of what a task or example has to offer. Gibson (1977) introduced the term *affordances* to refer to the uses perceived for an object by a potential user. His focus was on what the user perceives, whereas in teaching there are situations where teaching opportunities have been lost because a teacher has failed to perceive what an example offers, and yet that potential was present. This situation occurred in a probability game discussed by Chick and Baker (2005; see also Chick, 2007), where a teacher did not recognise the opportunity that the game afforded to consider sample space. The term *potential affordances* will thus be used to refer to the opportunities that are inherent in a task or example, whether perceived or not. To add to the complexity, the unpredictable nature of the classroom milieu can impact on whether or not an example's potential is realised. Watson suggests that there is "a complex interplay between what *could be* possible, what *is* possible, and what *is seen as* possible" (Watson, 2003, p.37).

Teachers use examples in different ways, suggesting the need for a categorisation scheme. Rowland and Zaslavsky (cited in Bills et al., 2006) propose two categories: examples that are used to illustrate a general principle and reasoning, and examples that are provided as exercises for practice. Rissland-Michener (1978) distinguishes four categories, possibly overlapping: (a) start-up examples that set up attention to the principle, (b) reference examples that are standard instances frequently referred to in the general theory, (c) model examples that show the typicality of a situation, and (d) counterexamples that show conditions under which the general principle might not apply. Drawing on these ideas—and clarifying who is most actively engaged in working through the example and whether the emphasis is procedural or conceptual—we propose a first level of classification that distinguishes the example according to whether it is a teacher-directed illustration (perhaps with exposition), a student task or exercise, or a past exercise that is then used by the teacher to provide further modelling or explanation for the class. Within the first and last categories we distinguish between whether the focus is procedural or conceptual, and also note if the example is a non-example or counterexample that sheds light on what the general principle is by showing what it is not. Within the student task category we further categorise according to whether the exercise is a routine application, establishes some cognitive conflict intended to highlight extra aspects of the general principle, or a non-routine task that extends understanding. This categorisation scheme is presented in Table 2 in the methodology section.

Pedagogical Content Knowledge

The foundation for many teacher actions in the classroom—including the choice of examples that are used to illustrate ideas—is *pedagogical content knowledge* (PCK). This term, introduced by Shulman (1986), emphasised that discipline knowledge alone is insufficient for successful teaching. In particular, he highlighted that there is knowledge that is particularly salient for teaching, yet which is still inextricably linked to content as well. For example, a teacher needs to know what models and explanations support learning, but also ensure that such models and analogies effectively convey the content ideas. Similarly, it is essential for a teacher to understand typical student conceptions, and why these conceptions might be held, and this understanding does not rely on discipline knowledge alone. Teachers also need to be able to determine what makes a task complex or easy. In the twenty years since Shulman wrote, aspects of PCK have been studied by many researchers, some of whom have focused on PCK in general, whereas others have studied particular components. As examples of some of this work, we mention Askew, Brown, Rhodes, Johnson, and Wiliam (1997), who drew attention to the importance of knowledge of connections among and within topics; Ball (2000), who highlighted the importance to teaching of being able to deconstruct knowledge into its key components; and Leinhardt, Putnam, Stein, and Baxter (1991), who emphasised the knowledge of representations and models. The importance of content knowledge has not been de-emphasised in this research (e.g., Kahan, Cooper, & Bethea, 2003), with Ma's 1999 work bringing to the fore the idea of Profound Understanding of Fundamental Mathematics (PUFM). Her work highlighted that there is important depth to seemingly basic concepts and that without this profound understanding teacher effectiveness is reduced. Marks (1990) provided further discussion on the intertwined nature of pedagogical knowledge and content knowledge, and Lampert (2001) has illustrated the complexity of applying this knowledge in the classroom. Based on this work, and that of other researchers, Chick, Baker, Pham, and Cheng (2006) developed a framework for pedagogical content knowledge (see Table 1). The framework attempts to identify the key components of PCK, how they are evident in teaching, and the degree to which both pedagogical and content knowledge are intertwined. The framework has been used to analyse teachers' responses to content and pedagogical questions on a questionnaire and follow-up interview (see Chick, Baker, et al., 2006; Chick & Harris, 2007; Chick, Pham, & Baker, 2006), and has begun to be used for classroom analysis (see Chick, 2007).

To illustrate the role of PCK in teaching, and to provide background for the context of the present study, consider the mathematical topic of ratios as taught in upper primary school. To introduce the topic, most teachers will pick an illustrative example of a context where ratio is important. Such an example may involve a discrete representation—such as comparing the number of boys to girls in the classroom—or a continuous representation—such as comparing the areas of shaded and unshaded regions. The choice made here affects what can be done at later stages of the lesson. The discrete representation, for instance, makes it difficult to talk about ratios involving fractional quantities. This knowledge of different models and what they offer is essential, and being able to evaluate the model's *epistemic fidelity*—that is, the capacity of the model to represent the mathematical attributes of the concept effectively—is vital for the teacher (see Stacey, Helme, Archer, & Condon, 2001, for a discussion and further illustration of this concept). To successfully teach about ratio the teacher must have the Profound Understanding of Fundamental Mathematics to know how ratios relate to fraction understanding (e.g., the idea of equivalence), to understand that it is important to distinguish what is being compared (e.g., part:part or part:whole), and to realise that multiplicative thinking rather than additive thinking is required. The teacher must have the knowledge of student thinking to recognise that some students may have difficulty identifying the components being compared, or that they may respond to a question with a particular incorrect answer because they are thinking additively.

Turning to the more specific issue of choosing and using examples, it seems that all aspects of PCK have the potential to be involved. There are, however, some aspects that seem particularly significant. Clearly the underlying content knowledge—such as PUFM and knowledge of connections and representations—provides the foundation for the teachers' work. Knowledge of student thinking—both current and anticipated, together with knowledge of likely misconceptions—is essential for selecting examples that help students progress in their understanding. Linked to this is the capacity to assess the cognitive demand of a task, so that assigned problems are achievable but challenging.

Table 1.
A Framework for Pedagogical Content Knowledge (after Chick, Baker, et al., 2006).

PCK Category	*Evident when the teacher …*
Clearly PCK	
Teaching Strategies	Discusses or uses general or specific strategies or approaches for teaching a mathematical concept or skill
Student Thinking	Discusses or addresses student ways of thinking about a concept, or recognises typical levels of understanding
Student Thinking - Misconceptions	Discusses or addresses student misconceptions about a concept
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Explanations	Explains a topic, concept or procedure
Knowledge of Examples	Uses an example that highlights a concept or procedure
Knowledge of Resources	Discusses/uses resources available to support teaching
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics (PUFM)	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a mathematical problem
Pedagogical Knowledge in a Content Context	
Goals for Learning	Describes a goal for students' learning
Getting and Maintaining Student Focus	Discusses or uses strategies for engaging students
Classroom Techniques	Discusses or uses generic classroom practices

The Current Study

Although there has been work on PCK and on examples, there has been little work looking more closely at the links between the two. We have argued that PCK influences example choice, so we would expect that example choice and use provides evidence for teachers' PCK. The present exploratory study is a subset of a larger study into the mathematical PCK held by upper primary teachers, and focuses on the topic of ratio. We will consider some of the examples chosen by two teachers as they endeavoured to teach early ratio understanding. The aim is to answer, perhaps only partially, the following questions:

1. What evidence of PCK can we gain by considering the teachers' choice of examples?
2. What examples do these teachers use for teaching ratio?

3. What affordances were present in the examples chosen and were good outcomes achieved from them?

Methodology

The data for this study came from the ARC-funded Mathematical Pedagogical Content Knowledge project, which involved fourteen Grade 5 and 6 teachers. A central focus of the project was to examine how teachers' PCK is enacted in the classroom. Initial data about a variety of aspects of teachers' PCK and mathematical content understanding were collected via a questionnaire and follow-up interview. The classroom data were collected by observing and video-taping pairs of lessons for each teacher, with up to four such lesson pairs being studied. The lessons within a pair were on the same topic, chosen by the teacher, and (usually) conducted consecutively. The video-camera followed the teacher, and audio was recorded via a wireless microphone worn by the teacher. Field notes were also made. After each pair of lessons had been conducted, the research assistant and the author interviewed the teacher about the lessons, allowing discussion of what happened in the lesson went compared to what had been planned, including critical learning incidents, and follow-up plans.

Two teachers conducted lessons with a focus on ratios. Both teachers were teaching Grade 6 classes, but in different schools. Clare had trained as a primary teacher through a one-year graduate program and had about five years' classroom experience. Hilary began her career as a secondary science teacher, but was now in her second year of teaching in the primary school. In both cases the studied lessons were their students' first lessons on ratio that year.

The teachers' video-taped lessons were subjected to a "content analysis" approach (Bryman, 2004), in which the definition of example (as given earlier) was used to identify instances of example use in the lessons. The examples were then categorized by the second author according to the framework in Table 2. For this paper, several illustrative cases were selected to highlight the different ways in which tasks were used, the affordances they offered, and the PCK involved. The purposeful selection of these cases makes them what Bryman (2004, p. 51) calls *exemplifying cases*. These are used to compare and contrast the approaches of the two teachers, with the intention not to make judgements but to examine the important but complex role of PCK in teaching with examples. It is recognised that an outside observer, repeatedly viewing a video-tape, has greater opportunity to identify alternative options that might have been more effective for teachers to use, but our intention is to shed light on the critical issues for teachers, not their choices per se.

Table 2.
A Framework for Categorising Examples.

Teacher Directed
Teacher-Directed Example — Illustrative/expository with conceptual emphasis
Teacher-Directed Example — Illustrative/expository with procedural emphasis
Teacher-Directed Non-example/Counterexample
Student Task
Student Task/Exercise — Routine application
Student Task/Exercise — Routine but provides cognitive conflict
Student Task — Non-routine task that extends/deepens understanding
Past exercises further used
Past exercises now used by teacher (or student) to model to the class — with conceptual emphasis
Past exercises now used for teacher (or student) to model to the class — with procedural emphasis

Results and Discussion

This section presents and discusses some of the examples used by the two teachers in their lessons on ratio. We will particularly focus on the pedagogical implications of the examples—such as the affordances offered, the way in which they were utilised, and the PCK evident or absent but needed. The examples are presented approximately in chronological order, beginning with the way that teachers introduced the topic.

Introducing the Topic of Ratio

Hilary's Lesson. Hilary began her lesson by asking the class if anyone knew what a ratio was, with the students' responses suggesting that some had heard the term but had little idea about its meaning. One student used the expression "a ratio of one to two" but could not illustrate its meaning. Hilary then explained that ratio is associated with fractions or proportion, and is used to show the amounts that comprise a whole. Her explanation was, at this stage, imprecise and given without an illustrative example. She then invited ten students to stand out the front, highlighted that the ten was the whole, and asked students to determine what proportion had light coloured hair and what proportion had dark coloured hair. She showed students how to write this as 3:7 and emphasised that 3+7 gives ten, the total in the group. She had students rearrange themselves to show the ratio of boys to girls, which turned out to be 5:5, before asking for another arrangement that turned out to be 3:7 again. On the basis of these examples Hilary then gave the students some notes about ratio.

Hilary used three different examples based in the same context: a discrete collection that she divided into two groups in three different ways. She used the examples to demonstrate the notation of ratio and how to say it, and to remind students about the whole and that ratio compares two numbers. Her emphasis during the introductory exposition appeared to be partly conceptual, but with a strongly procedural emphasis in the discussion of the way to write and say ratios. Her notes on the board for students to copy included an extra example, 1:5, which initially had no physical context, and which her notes suggested could also be written as $^1/_5$. This was done without comment and was problematic because additional explanation is necessary. Her indication that 1:5 and $^1/_5$ are the same suggest that although she recognised that fractions and ratios are inextricably linked, her PUFM and capacity to deconstruct knowledge to its key components were insufficient to identify the distinctions between the way parts and wholes are used in the two contexts. She later illustrated the 1:5 example in the context of making cordial, where she highlighted that one part of cordial and five parts of water should be used, to give a total of six equal parts. She also clarified that the actual size or amount of these equal parts did not matter, provided all parts are equal, and emphasised that the order of the numbers in the ratio matters, thus indicating an appreciation of some of the key components for ratio. Despite these additional clarifications, however, she did not address the connection to the fraction $^1/_5$.

Clare's Lesson. Clare began by showing the students a 2cm×3cm rectangle and reviewing the definition of area and perimeter, highlighting meaning and units. She then demonstrated how to colour in the rectangle in such a way that for every square that was coloured in red, two squares had to be coloured in blue, before having a student repeat this process for another 2cm×3cm rectangle. She asked the students how many squares were coloured red and how many were blue, but before this had been answered one student pointed out that $^1/_3$ of the rectangle was red and $^2/_3$ was blue. This unexpected response allowed Clare to explore the connection between fractions and the situation that they had, highlighting that the $^1/_3$ came from the fact that $^{2}/_{6}$ of the squares in the rectangle had been coloured red. After showing students that they could also colour half squares while still achieving one red colouring for every two blue, and demonstrating such an example, she asked students to find the different colourings of the 2cm×3cm rectangle using the "one red for every two blue" scheme. As they started work she drew from them the need to work systematically, suggesting that they start with whole square colourings first, and attend to the different possible positions of the red coloured squares. After allowing students to explore the problem for about 15 minutes she had students talk about how they had worked through all the possible arrangements of red and blue colourings, incorporating some discussion about how equivalent arrangements can arise by "flipping" (reflecting) arrangements already found. She concluded her use of this example by emphasising to students that although they had produced many different arrangements, the area of red in all cases was $2cm^2$ and the area of blue was $4cm^2$.

Clare did not mention ratio at all during the first 25 minutes of the lesson; the overt emphasis seemed to be on area, working systematically, and then, briefly, ideas of symmetry. However, her choice of the 2cm×3cm rectangle, and the simple proportion "one red to two blue" allowed students to consider the area, problem solving, and symmetry ideas—with fractions receiving some consideration as well—while implicitly building a foundation for talking about ratio. It was only after this exploration of a single example that she actually defined ratio, using the 2 red to 4 blue idea, helping students to see the connection to fractions, identifying the connection between the parts and the whole, getting students to simplify the ratio 2:4 to the "basic" ratio 1:2, and linking this back to her original colouring instruction to colour 1 red and 2 blue. The example used—the 2cm×3cm rectangle and the ratio1:2—was used for teacher demonstration with a conceptual focus, but was also used as a student task, and the focus was on conceptual ideas rather than procedural ones. In addition, Clare demonstrated strong PCK in being able to use the example to illustrate connections among so many mathematical topics.

Teacher Demonstrations

Hilary's Lesson. Having defined ratio and having students work on some ratio situations (discussed in the next section) Hilary concluded the first lesson by discussing some examples she had written on the board. These included "Does 4:18 equal 1:4?", "Does 15:25 equal 3:5?", and then four simplifications: 2:8, 4:6, 36:72, and 9:27:18. The fourth problem here involved three parts, a situation the students had not encountered. At this point she showed a physical representation and example of the three-parts ratio situation involving ten pieces of duplo, eight textas and four ping-ping balls, but did not talk about how simplifying is done in this case. When referring back to the assigned three-parts problem on the board Hilary said that this might represent the colours present in a box of Smarties. The duplo/textas/ping-pong balls example was a contrived context for the three-parts situation, whereas the Smarties situation was certainly appropriate. Her choices for the numbers in all these examples were appropriate for simplifying, although when discussing the simplification process she did not use the term "factors", instead talking about numbers that "go in".

After a review at the beginning of the second lesson, Hilary, with guided responses from the students, worked through two more simplification examples on the board, 5:20 and 9:12. Following this she discussed the more general idea of equivalent ratios, including those in which the numbers become larger than in the original ratio. She used the 5:20 example to illustrate that doubling both numbers, just as in fractions, produces an equivalent ratio. When she asked the students to suggest how they could produce an equivalent ratio for 9:12 the first suggestion was to double the numbers (as in the earlier example), and when Hilary asked for a second answer that did not involve doubling, one student suggested multiplying by ten. Hilary then said that any number could be used for multiplying but did not ask for an additional example to show this. While there was nothing wrong mathematically with obtaining equivalent fractions by doubling or multiplying by ten, there is an interesting point to be made about the way these two cases were presented. In the doubling case there was actually no emphasis on the fact that doubling means multiplication by two (indeed, it is known that many students perceive "doubling" as an operation in its own right, not necessarily related to multiplication at all), and in the multiplying by ten case the new numbers were actually obtained and explained by "adding a zero on the end", which again obscures the actual multiplication aspect. An extra example in which the multiplication was made more explicit may have been useful. Recognition of this rather subtle point requires PUFM, and the ability to identify the absence of explicitly expressed multiplication in the two examples given.

Clare's Lesson. Some of Clare's demonstrations were associated with discussions of student tasks, and are described in that section. At the beginning of the second lesson Clare recapped the example that they had done in the previous lesson with colouring a rectangle of area 6cm^2 using a ratio of two red to six blue. She recorded 2:4 on the board and elicited the simpler ratio 1:2 from students, which she wrote underneath. She then asked students what to do to change one ratio into the other, drawing arrows from the 2 and 4 of the first ratio to the 1 and 2 respectively in the second. She explicitly recorded "÷2" next to each of the two arrows, and made the connection to equivalent fractions. This explicit discussion and recording of the operation as division by two (rather than halving) is in contrast to the example of Hilary earlier. Clare also highlighted that the basic ratio 1:2 did not tell them how many squares had been involved in the original situation, and made appropriate connections to fractions, clarifying the part-whole relationships.

Clare then set up a table in which she started to make explicit connections between the parts, the total, and equivalent ratios. Still using the red/blue colouring in rectangles as the context, she headed the table with "Red", "Blue", and "Total". As her first example, she picked a ratio of 2:3, recording 2 and 3 in the Red and Blue columns respectively, and put 5 in the Total column. She then asked students to consider what would happen if she now had a rectangle with 15 squares, and wrote 15 in the Total column. She asked what had to be done to the 5 to turn it into 15, and when students responded "×3", she wrote this down next to an arrow between the 5 and 15, and then elicited from the students that the same operation needed to be applied to the parts in order to get the equivalent ratio 6:9. She then worked through a "simplify" example, set up in the same table structure, and emphasised the role of the common factor. This demonstrated clear attention to connections among topics, and the importance of using representations that facilitate understanding and calculation (in this case, the three-column table with the arrows and marked operations).

In preparation for having students complete some problems on a worksheet, Clare worked through an example that she seemed to have invented herself: "Chris and Anna share $10 with a ratio 2:3. This means that for every $2 Chris receives, Anna will get $3. How much do they each get?" Clare demonstrated how to solve this, by careful questions that allowed the students to explain the steps, while she recorded the calculations using the three-column table. Emphasis was placed on the role of the multiplying factor. She continued the question-and-answer exemplification of the procedure by also asking "How much money do they each have if the total is $35?" and, finally, "If Chris gets $10 how much will Anna get?" She followed this up with another example involving mixing cordial and water in the ratio of 1:5. Again Clare considered both part and whole relationships, and used simple multiplying factors, in which the multiplication operation being emphasised. By keeping the numbers straightforward Clare allowed students to attend to the process rather than the computations.

Student Tasks with Ratios

Hilary's Lesson. After her introduction at the beginning of the first lesson Hilary then gave each pair of students a pre-prepared bundle of straws, and asked them to work out the ratio of red straws to blue straws, asserting that there were only four different ratios in the collections. She also asked the students to simplify the ratio obtained. Hilary had chosen the examples 5:10, 12:4, 8:6 and 15:3, all of which can be simplified as their common factors are easy to identify. Although each pair had only one example to work on at this stage, this was a routine application of what Hilary had explained earlier and their previous work simplifying fractions. The cognitive demand of the task was not too difficult for the students and allowed Hilary to focus on the notation and the idea of simplifying the ratio. Despite this, however, there was no discussion of the conceptual connection between the simplified ratio and the non-simplified one.

Hilary then had two (sometimes three) pairs combine their straw bundles and obtain ratios for the new groupings, then simplify if possible. In many cases, the resulting ratios could not be simplified, and Hilary was able to discuss with students that there was nothing that would go into both numbers apart from one. For these examples, Hilary could not have predicted which specific numbers would arise, but she had anticipated that some would not simplify. PCK is evident in good choices of factorisable numbers for the ratios in the original bundles, and allowing cases that would not simplify when the bundles were combined.

In the second lesson, after introductory discussion of equivalent ratios, Hilary asked students to consider the ratios 1:10, 13:13, 2:3, and 6:9:12, and, in each case, devise a real life situation that could be represented by that ratio, indicate how many parts in total make up the ratio, and find an equivalent ratio. When Hilary was talking to students as they worked, one of the students needed assistance with producing an equivalent ratio for 13:13. Hilary asked if it simplified down, and then attempted to clarify this by asking if anything went into 13. It seemed that Hilary was too mindful of 13's prime status, because she acknowledged the student's response that nothing seemed to go into 13, and then, consequently, had students construct an equivalent ratio involving larger numbers because she had failed to see that dividing both parts of 13:13 by 13 would allow simplification to 1:1. Later another student asked about simplifying the ratio 2:3 by dividing by two, and Hilary did not permit the possibility of expressing the ratio as 1:1.5 because, in Hilary's words, "two does not go into three evenly". In both these instances, Hilary has correct understanding about primeness and the divisibility of three by two, but this knowledge was incompletely applied in the ratio context. Interestingly, when Hilary had students contribute their answers in a class discussion when the problems had been completed, one student proposed 2:2 as an equivalent ratio for 13:13, explaining that he had reduced it to 1:1 and then doubled. She accepted this explanation, adding that if you have half the people in one group and half the people in the other group, then it doesn't matter if there are 100 or 99 people in the group. She wrote down 1:1 as the equivalent, but then was incorrect in explaining that this was permissible even though the two did not go into 13 and they had not been able to simplify by finding a number that went into both 13s. It should be noted that Hilary did not pursue alternative answers from the students, with the exception of having both a simplified and doubled version of 6:9:12, and so missed an opportunity that the activity afforded and which might have added to students' understanding of equivalent ratios.

Hilary concluded her second lesson with a worksheet, sourced from the web, claiming that the first three questions should be relatively straightforward, but that the last five would require the students to think. The first two questions were routine applications of applying ratio notation, and the only difficulty that some students had was in attending to the order. The remaining problems required very careful application of proportional reasoning, to parts and to the whole, in real-world contexts. The first was relatively straightforward, asking that if the ratio of cups of water to cups of oatmeal is 3:1 then how many cups of oatmeal are required if 12 cups of water are used. Most of the students seemed to manage this on their own, despite not having encountered any examples in which a specific equivalent ratio needed to be obtained by identifying the appropriate multiplying factor, but a few required assistance. Difficulties were more evident in the next example, which stated that the ratio of the number of boys to girls in a swimming club is 7:4, and asked how many boys are there if there are 121 children in the club. There seem to be two likely sources of difficulty in this problem, the first associated with the need to attend to the total and find the scaling factor from that, and the second arising because knowledge of multiplication number facts for 11 is required. It appeared that few if any students attempted any of the examples after this, as the lesson ended. There was no time for follow-up at the end of the lesson; Hilary concluded the lesson by saying that they had enough knowledge to do the problems and that they would consider them in a following lesson. Hilary was correct in her assessment of the cognitive load of the examples in one sense, but appeared to have misjudged her students' capacity to tackle the harder ones without modelling of similar tasks. There had been much teacher-directed guidance on the earlier basic problems; it was curious that there had not been similar demonstrations of application problems in advance of requiring students to attempt such problems.

Clare's Lesson. After Clare's extensive introduction with the 2cm×3cm rectangle she asked students to consider a 5cm×3cm rectangle (she corrected this after initially saying a 5cm×2cm rectangle), and asked them to colour two squares red for every three blue, with students correctly responding that this meant a ratio of 2:3. She recognised the scope of this task when she acknowledged to students that they were unlikely to get to a consideration of the half-square colourings that they had allowed in the earlier example. Once again the example only revealed one equivalent ratio for 2:3, namely 6:9, but showed this ratio in many different arrangements and the problem-solving involved in systematically enumerating the possibilities is worthwhile. Students worked on this for about ten minutes, and then Clare demonstrated on the board that colouring would always involve six red and nine blue, and highlighted that they could check the correctness because 6+9=15 and 5×3=15. She also explained that the fraction for the proportion of reds involved is $^6/_{15}$ and not $^6/_9$, by highlighting what constituted the whole.

Clare concluded her first lesson with another single example that was an extended task. She asked students to find rectangles that could be coloured in the ratio of 1:3, with one red to three blue. A student suggested an area of 4cm^2 as one possibility, and then Clare got them to search for other rectangles that could be coloured in the same ratio. Implicit in this was a requirement to colour whole squares, thus requiring rectangles with an area that is a multiple of 4. From the initial instructions it was not immediately evident that rectangles other than those with an area of 4cm^2 were required, but Clare established this with individual students as she moved around the room. Both this and the "whole number of squares" requirement were clarified in a later whole class discussion in which Clare had the students recognise that they needed to be searching for rectangles with area divisible by 4. This example was an "inverse" problem from the previous tasks where the area had been given, and allowed students to see more equivalent ratios, see the role of the total of the two parts in the ratio, and explore another enumeration task. Class discussion at the end of the lesson revealed the equivalent ratios 1:3, 2:6, 3:9, and so on, from all the different rectangles. One student then highlighted that finding a number that goes into both parts of the ratio will allow simplification of the ratio, and another student highlighted that it would be possible to show 4:12 within the 4cm^2 rectangle. This example afforded the development of many important ideas for ratio, even though it considered only one ratio value. The connections to other topics afforded by this example are again evident.

During the second lesson Clare had several containers containing different numbers of different coloured blocks. She required students to count the blocks, determine the ratio as counted, and then simplify to get the basic ratio, recording this in three columns as demonstrated on the board earlier. Students had to do this for as many containers as possible. Clare had chosen the following ratios for the blocks in the containers—21:14, 32:8, 20:40, 15:6, 21:7, 15:25, 10:15, 6:18, 14:8, and 20:25. Simplifying required students to identify the required factors and carry out the division, and the range of different factors chosen gives students ample opportunity to practice ratio calculation. The only case missing that may have been worth including was a 1:1 situation. When discussing the answers with the class there was good attention to the operations involved. Less care was taken in maintaining the order of the parts (e.g., 1:4 was accepted as the basic ratio for 32:8), although this may have been because some students may have recorded the original ratio differently (e.g., as 8:32) since the choice for the initial order was arbitrary. In this case the task, as designed, had a limitation on what it afforded because of this issue with order.

For the second half of Clare's lesson, following her demonstration of worded examples, she gave students a worksheet, sourced from a book, of worded problems. The second example concerned a concert attended by 400 people in which the ratio of children to adults was 7:3, and asked students to determine how children attended. After students had worked on this Clare emphasised to the whole class the importance of setting out their working in the way she had explained. A student also highlighted that the answer could be checked by adding the obtained number of children and adults to see if 400 could be obtained, and another student explained that instead of going directly from the total of 10 to the total of 400 by multiplying by 40, he had scaled to a total of 100 and then added this four times. Clare recognised that this approach was correct, thus demonstrating her knowledge of different methods of solution, and that the scope of the example afforded such different approaches. The next example was worded as "one in every thirty raffle tickets wins a prize"; this required more care for students to identify the categories defining the parts (winning tickets and losing tickets) and the total (30). The following question about chocolates with soft centres had a similar structure. Clare used this to highlight the importance of attending to the wording. One of the final tasks involved concrete mixing in which three parts were involved. Although Clare had not chosen the values in the examples on the worksheet, she had chosen the set of examples by choosing the worksheet, and seemed to have picked appropriately for the capacity of her class and also provided a sufficient foundation for students to tackle it.

Other Examples

Hilary's Lesson. During the straw bundles exercise one of the students asked Hilary where ratios are actually used, and Hilary gave the example of using different proportions of ingredients for cooking or making a cup of coffee. This illustration demonstrates that Hilary has an understanding of the purpose of this particular area of content.

At the beginning of the second lesson Hilary asked the students to give *her* examples of ratios. One student suggested 1:5, in the context of comparing hair colours; a second student suggested 5:10, attempted to change it to 6:2 but was encouraged to keep it as 5:10. This student proposed that this might be the ratio of lights to fans on the ceiling, which, as became apparent later, was actually the 6:2 that he had put forward in preference to the initial 5:10. A third student suggested 3:6 as an example, and used cordial mixing as the context. After these examples had been given, Hilary asked for the simplified ratios, but again did not draw out the connection between the simplified version of the ratio and the original, even when it was shown that 5:10 and 3:6 simplified to the same ratios.

Clare's Lesson. In Clare's second lesson a student asked if a ratio could consist of three numbers, and Clare responded with the example of having two orange, three brown, and four green squares, which would be written as 2:3:4.

After the students had worked through the containers of blocks problems, about half-way through the second lesson, Clare highlighted that ratios are largely meaningless on their own, and elicited several real-world examples and provided some of her own, including the ratio of teachers to students on an excursion, thus demonstrating knowledge of the purpose of content.

In order to help students to unravel the language of "one in thirty" for the raffle ticket example, Clare made up an analogical example about one out of the 23 people in the class learning violin, allowing her to highlight that the total was 23. This example was enough to trigger understanding for some students; in other cases, she asked additional guiding questions.

At the conclusion of the second lesson Clare invited students to select an example that had "stumped them", and this was the task that she focused on. She tried to get students to identify what it was that had caused difficulty, allowing them to identify the likely misconception of thinking that the ratio of winning tickets to losing tickets is 1:30 rather than 1:29. Clare was not only aware of this misconception herself but seemed to think it important to allow students to explicitly recognise the possibility of it arising.

Conclusions

Both teachers seemed to cover the same material and reach nearly the same endpoint over the course of their two lessons. We note that Clare's lessons were longer than Hilary's, which provided her with the time to cover more material, and this may explain why Clare got further with the application problems on her worksheet. Interestingly, the final worksheets used by the teachers were quite similar in their content and reflect, perhaps, similar example choices. What was striking was the preparation done leading into the worksheets. Closer examination of the earlier examples and demonstrations suggests that Clare may have provided her students with a better foundation for tackling the worksheet tasks. Certainly she had explicitly modelled the way in which to solve those problems, and yet students could not just mimic her procedure since some of the examples on the worksheet required careful identification of the parts and the whole.

One of the most striking things about Clare's lessons, evident here and in other lessons (see Baker & Chick, 2006; Chick, 2007), was her capacity to make connections fluently among a range of mathematical topics. In the present lessons she made strong use of area understanding (and referred to perimeter in passing), was careful in establishing the links to fractions, and used correct terminology such as "factor". Hilary certainly appeared to understand the content, but was not explicit about the connection between a ratio and its simplified form. There was no explanation of the meaning, in a conceptual sense, of saying that the ratio 1:4 is the same as 2:8. Clare achieved this connection more successfully by beginning with the simplified ratio but having students use it in a situation that was going to result in larger quantities of units (i.e., she used the ratio 1:2 but in a situation that forced students to produce something that could naturally be considered as 2:4). Clare's choice of context for her example afforded this development of understanding, whereas Hilary's did not: there was no inherent reason, with the straws, to consider why you might think about 10:15 as 2:3. Similarly Clare's initial work in having students build up an equivalent ratio from the simpler one, and allowing students to see multiple configurations of square colourings all of which show 2:4, may have made it much easier for students to appreciate the idea of simplifying the ratio.

The situations described in this paper highlight the difficulty of selecting appropriate examples, and using them effectively to illustrate general principles. The need to choose suitable representations is particularly important. The discrete representations of ratios apparent in the groups of people and the straw bundles used in Hilary's class probably restrict full understanding of ratio, when compared to the continuous area model used in Clare's class, especially as she allowed students to colour half squares and to consider quarters as well. The sequencing of examples was also important, with Clare building up non-simplified ratios before considering simplification and equivalence, and she also tried to ensure that students were prepared for the problems on the worksheet.

Returning to the research questions for this paper, it seems that we can learn much about teachers' PCK by considering the examples they chose and the way they used them in the classroom. While we have not attempted to exhaustively examine the PCK evident in what these teachers did, the situations considered certainly show the presence of aspects of PCK being enacted in teaching, and in some cases its absence can also be detected and be seen to be problematic. In terms of the examples chosen for ratio, in many respects they were very similar for both teachers, with one noticeable difference being the use of discrete and continuous contexts as initial scenarios for establishing the meaning of ratio. Clare's extensive exploration of the two rectangle colouring problems meant that the students did not appear to consider very many examples, but this seemed to provide a strong foundation for later work. She also chose better examples to give prior to assigning students the worksheet. Finally, Clare seemed to maximise the affordances of an example, as seen by the way she drew so much out of the area-colouring problems, although the omission of a 1:1 example from the containers of blocks was a small limitation. Hilary generally used her examples effectively, but did not make connections to the same degree as Clare. Hilary could have utilised a greater variety of values for the factors involved in the initial simplification problem, and it was unfortunate that she did not take advantage of the opportunity to consider other possible equivalent ratios in the four examples given in the second lesson.

With the assumption, not directly examined here, that examples facilitate learning, it may well be time to help teachers think more explicitly about the affordances in an example. To do this successfully certainly requires PCK, particularly an understanding of the underlying mathematics and the connections among topics, plus an understanding of cognitive demand and the appropriateness of the example for the concept. Having teachers ask themselves questions about the examples that they are considering for use may be useful. Such questions might include: "What is the mathematical concept that I want to exemplify with this example? What do I actually want this example to do? Can this particular version of the example do what I want? Does it do it clearly enough or are there things that interfere? Can I make the example do its exemplifying job better? Is there more that the example offers in addition to the content of primary interest? Do I want to capitalise on that? How can I implement this example in the classroom so that the general principles are highlighted? Do I need to make the example harder or easier? How do I do this; what do I change? What changes within the example as a consequence? Have I lost or gained any of its features?" If we can do this, then perhaps examples will become better exemplifiers of general principles.

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