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Assessing Multiplicative Thinking Using Rich Tasks

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Recent research has identified multiplicative thinking as a major barrier to students' mathematical progress in the middle years of schooling. New approaches to assessment are needed to inform more targeted teaching and improve student numeracy outcomes at this level. The purpose of this paper is to describe the processes involved in developing a range of rich assessment tasks to evaluate students' multiplicative thinking in Years 4 to 8. Key concepts used as a basis for designing the tasks, as well as the tasks, scoring rubrics, and assessment protocols used to collect data from just under 3500 students will be illustrated, together with sample student responses.

Background:

The material presented in this paper is drawn from a recently completed research project aimed at scaffolding numeracy learning in the middle years of schooling[1]. The project was prompted by the results of an earlier study which indicated that many students in Years 5 to 9 have difficulty with what might broadly be described as multiplicative thinking. That is, thinking that is characterised by

(i) a capacity to work flexibly and efficiently with an extended range of numbers (for example, larger whole numbers, decimals, common fractions, and/or per cent);

(ii) an ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion; and

(iii) the means to communicate this effectively in a variety of ways (for example, words, diagrams, symbolic expressions, and written algorithms).

In particular, the results suggest that while most students are able to solve multiplication problems involving relatively small whole numbers, they rely on additive strategies to solve more complex multiplicative problems involving larger whole numbers, rational numbers, and/or situations not easily modelled in terms of equal groups (Siemon & Virgona, 2001). This suggests that the transition from additive to multiplicative thinking is nowhere near as smooth or as straightforward as most curriculum documents seem to imply, and that access to multiplicative thinking, as it is described here, represents a real and persistent barrier to many students' mathematical progress in the middle years of schooling.

This observation is supported by the literature more generally. For example, there is a considerable body of research pointing to the difficulties students experience with multiplication and division (Mulligan & Mitchelmore, 1997; Anghileri, 1999), and the relatively long period of time needed to develop these ideas (Clark & Kamii, 1996; Sullivan, Clarke, Cheeseman & Mulligan, 2001). Student's difficulties with rational number and proportional reasoning have also been well documented (for example, Hart, 1981; Harel & Confrey, 1994; Lamon, 1996; Baturo, 1997; Misailidou & Williams, 2003). Moreover, there is a growing body of research documenting the link between multiplicative thinking and rational number ideas (Harel & Confrey, 1994; Baturo, 1997); multiplicative thinking and spatial ideas (Battista, 1999), and the importance of both as a basis for understanding algebra (Gray & Tall, 1994). While this work contributes to a better understanding of the 'big ideas' involved, very little is specifically concerned with how these ideas relate to one another and which aspects might be needed when, to support new learning both within and

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[1] *Scaffolding numeracy in the Middle Years – An investigation of a new assessment-guided approach to teaching mathematics using authentic assessment* tasks 2003-2006, an ARC Linkage Research Project awarded to RMIT University in collaboration with the Victorian Department of Education & Training, and the Tasmanian Department of Education

between these different domains of multiplicative thinking. Moreover, very little of this work appears to be represented in a form and language that is accessible to teachers or directly translates to practice in the middle years of schooling.

Simon's (1995) idea of constructing hypothetical learning trajectories (HLTs) as mini-theories of student learning in particular domains appears to offer a useful approach to these problems as they provide an accessible framework for identifying where students 'are at' and offer starting points for teaching.

In Australia, learning trajectories have tended to take the form of *learning and assessment frameworks* which have been developed and validated in terms of a number of discrete domains such as counting, place-value, and addition in the early years of schooling (Clarke, Sullivan, Cheeseman & Clarke, 2000). These are typically developed on the basis of large-scale interview data and represented in a form that is accessible to teachers. For example, as a part of the Victorian *Early Numeracy Research Project* (ENRP) the following growth points for multiplication and division were identified:

0 Not apparent - Not yet able to create and count the total of several small groups

1 Counting group items as ones – To find the total in a multiple group situation refers to individual items only

2 Modelling multiplication and division (all objects perceived) – Models all objects to solve multiplicative and sharing situations

3 Abstracting multiplication and division – solves multiplication and division problems where objects are not all modelled or perceived

4 Basic, derived and intuitive strategies for multiplication – Can solve a range of multiplication problems using strategies such as commutativity, skip counting, and building up from known facts

5 Basic, derived and intuitive strategies for division - Can solve a range of division problems using strategies such as fact families and building up from known facts

6 Extending and applying multiplication and division – Can solve a range of multiplication and division problems (including multi-digit numbers) in practical contexts (Sullivan, Clarke, Cheeseman, & Mulligan, 2001)

The ENRP also found that where teachers were supported to identify and interpret student learning needs in terms of the these frameworks, they were more informed about where to start teaching, and better able to scaffold their students' mathematical learning (Clarke, 2001). While learning and assessment frameworks for multiplication and division have been developed for the early years of schooling, the evidence suggests that very few students in Years P to 3 are at the point of abstracting multiplicative thinking, that is, able to work confidently and efficiently with multiplicative thinking in the absence of physical models (Mulligan & Mitchelmore, 1997; Sullivan et al., 2001). This suggests that developing a learning and assessment framework for the key ideas involved in multiplicative thinking that goes beyond these early stages that teachers can use to identify student learning needs and plan targeted teaching interventions, is likely to contribute to enhanced learning outcomes for students in the middle years.

The Project

As previously indicated, the research reported in this paper was a key part of a larger study concerned with documenting the development of multiplicative thinking which is known to be a major barrier to students' mathematical progress in the middle years. A major component of the research study was the identification of an evidence-based learning and assessment framework which could be used to support a more targeted approach to the development of multiplicative thinking in the middle years of schooling.

In order to achieve this, the study was designed in terms of three overlapping phases. Phase 1 was aimed at identifying a broad hypothetical learning trajectory (HLT) which would form the basis of the proposed learning and assessment framework for multiplicative thinking (LAF). Phase 2 involved the design, trial and subsequent use of a range of rich assessment tasks which were variously used at the beginning and end of the project to inform the development of the LAF. Phase 3 involved research school teachers and members of the research team in an eighteen month action research study that progressively explored a range of targeted teaching interventions aimed at scaffolding student learning in terms of the LAF.

Just over 1500 Year 4 to 8 students and their teachers from three research school clusters, each comprising three to six primary (K-6) schools and one secondary (7-12) school, were involved in Phases 2 and 3 of the project. A similar group of Year 4 to 8 students from three reference school clusters was involved in Phase 2 only.

Phase 1:

The initial HLT was derived from a synthesis of the research literature on students' understanding of multiplicative thinking, proportional reasoning, decimal place-value and rational number (see Table 1). It comprised nine 'levels' of increasingly complex ideas and strategies grouped together more on the basis of 'what seemed to go with what' than any real empirical evidence, although this was used where available.

Table 1. Derivation of Initial HLT (Draft Learning and Assessment Framework)

Hypothetical Learning Trajectory (SNMY, 2003)	Source
Recognises equal groups but not as a composite unit, counts group items by ones to check equality or find total, may lose track of count.	Steffe, 1994; Mulligan & Mitchelmore, 1997; Brown, 1981; Anghileri, 1989; Sullivan et al, 2001
Beginning to recognise numbers up to 5 as composite units, skip counts larger collections, shares small collections into equal groups, prefers to represent 2-digit numbers as ones.	Hart, 1981; Sullivan et al, 2001; Young-Loveridge, 2003
Regards 10 as composite unit, counts large collections efficiently but needs to see all groups, shares larger collections, recognises half and quarter, consolidating whole number place-value	Brown, 1981; Anghileri, 1989; Ross, 1989; Stanic & Killon, 198`9
Regards 10 as composite unit, counts large collections efficiently but needs to see all groups, shares larger collections, recognises half and quarter, consolidating whole number place-value	Brown, 1981; Anghileri, 1989; Ross, 1989; Stanic & Killon, 1989
Determines total given partially obscured array or region using repeated addition. Uses abbreviated methods for counting groups, eg, doubling or known addition facts.	Fischbein et al, 1985; Steffe, 1994; Sullivan et al, 2001; Mulligan & Mitchelmore, 1997
Determines total independently of models using known multiplication facts and/or mental strategies based on doubling, place-value and addition facts, recognises simple proper fractions, including tenths.	Hart, 1981; Anghileri, 1989; Steffe, 1994; Sullivan et al, 2001
Thinks of division in terms of multiplication, solves broader range of multiplication problems involving whole numbers using informal strategies and formal recording.	Hart et al, 1985; Quintero, 1986; Greer, 1992; Mulligan & Mitchelmore, 1997; Young-Loveridge, 2003
Applies 'fraction as operator' idea to solve problems involving 'simple' fractions and known percents, evidence of *area* idea for multiplication	Hart 1981; Greer, 1992; Simon, 1992; Callingham, 2003
Works confidently with ratios, rates, and percents, rename and compare fractions, recognise bi-directional multiplicative relationship between place-value parts, solve simple ration and proportion problems multiplicatively	Hart, 1981: Greer, 1992; Confrey, 1994; Baturo,1998; Miller & Fey, 2000
Uses appropriate representations, language and symbols to deal effectively with unfamiliar multiplicative situations including proportion problems involving rational number.	Hart et al. 1985; Greer, 1992; Clark & Kamii, 1996; Young-Loveridge, 2003

The development of multiplicative thinking requires that students construct and coordinate three aspects of multiplicative situations (groups of equal size, number of groups and the total amount), "in such a way that one of the composite units is distributed over the elements of the other composite unit" (Steffe, 1994, p.19).

Initially, multiplicative thinking develops as each element (equal groups, number of groups, total) is abstracted, that is, it can be dealt with as a mental object.

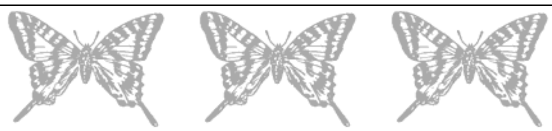
Modelled	Abstracted
Equal groups Makes all, counts all	Trusts the count, sees equal group as a composite unit
Number of groups Sees in terms of each group, counts all groups, 1 group, 2 groups, 3 groups …	Can deal with number of groups in terms of part-part-whole understanding, eg, 6 groups is 3 groups and 3 groups or 5 groups and 1 group
Total Arrived at by counting all, skip counting	Total seen as a composite of composites, eg, 18 is 2 nines, 9 twos, 6 threes, 3 sixes

While working with concrete models and representations is important in the early stages, a key aspect in crossing the 'abstracting barrier' appears to be the capacity to work with mental images and strategies based on doubling and known facts without physical objects. A critical step in this process appears to be the shift from counting groups to seeing the number of groups as a factor and generalising. That is, from counting groups, for example, 1 three, 2 threes. 3 threes, 4 threes, ..., to the consideration instead of 3 ones, 3 twos, 3 threes, 3 fours, 3 fives, …. This shifts the focus from the number *in each* group to the number *of* groups which supports more efficient strategies. In this case, 3 of anything, is double the group and one more group. Students then need to move from models and representations that work for whole number, to more general ideas accommodating rational number and algebra. These ideas include ratio, proportion, multiplicative comparison, multiplication of measures, and the use of intensive quantities. This process is complex and may take many years to achieve.

The initial HLT or Draft Learning and Assessment Framework was used to select, modify and/or design a range of rich tasks including two extended tasks (Callingham & Griffin, 2000; Siemon & Stephens, 2001). The tasks were trialled and either accepted, rejected or further modified on the basis of their accessibility to the cohort, discriminability, and perceived validity in terms of the constructs being assessed. Trial data were used to develop scoring rubrics and feedback from the trial teachers was used to modify the assessment protocol. Two examples are given below together with their scoring rubrics. *Butterfly House* (adapted from Kenney, Lindquist & Heffernan, 2002) was an extended task completed over 2 maths lessons and *Adventure Camp* was one of four shorter tasks which were completed in one maths lesson. Prior to attempting the tasks, students were asked to 'score' two sample student responses to a similar task to ensure that they understood what was meant by instructions such as "use as much mathematics as you can" or "show all your working and explain your answer in as much detail as possible".

Example of an Extended Task:

BUTTERFLY HOUSE…



Some children visited the Butterfly House at the Zoo.
They learnt that a butterfly is made up of 4 wings, one body and two feelers. While they were there, they made models and answered some questions.

For each question, explain your working and your answer, in as much detail as possible.

a. How many wings, bodies and feelers would be needed for 7 model butterflies?

_______________ wings

_______________ bodies

_______________ feelers

b. How many complete model butterflies could you make with 16 wings, 4 bodies and 8 feelers?

c. How many wings, bodies and feelers will be needed to make 98 model butterflies. **Show all your working and explain your answer in as much detail as possible.**

_______________ wings

_______________ bodies

_______________ feelers

d. How many complete model butterflies could you make with 29 wings, 8 bodies and 13 feelers? **Show all your working and explain your answer in as much detail as possible.**

e. To feed 2 butterflies the zoo needs 5 drops of nectar per day. How many drops would they need each day for 12 butterflies? **Show all your working and explain your answer in as much detail as possible.**

f. How many butterflies could you feed with 55 drops of nectar per day? **Show all your working and explain your answer in as much detail as possible.**

g. How many butterflies could you feed with 135 drops of nectar per day? **Show all your working and explain your answer in as much detail as possible.**

h. Model butterflies can be made with wings, grey, brown or black bodies and either long or short feelers. How many different model butterflies are possible? **Show all your working and explain your answer in as much detail as possible.**

i. In addition to either grey, brown or black bodies and either long or short feelers, model butterflies can also be made with either all yellow, all blue or all red wings. How many different model butterflies can be made now? **Show all your working and explain your answer in as much detail as possible.**

Scoring Rubrics for Butterfly House:

BUTTERFLY HOUSE …		
TASK:	**RESPONSE:**	***SCORE***
a.	No response or incorrect	**0**
	Correct (28 wings, 7 bodies, 14 feelers)	**1**
b.	No response or incorrect	**0**
	Correct (4 butterflies)	**1**
c.	No response or incorrect	**0**
	Partially correct with some indication of multiplicative thinking (eg, multiplication algorithm attempted), or correct but evidence of additive thinking, eg, 98+98+98+98	**1**
	All correct (392 wings, 98 bodies, 196 feelers) with evidence of multiplicative thinking, eg, algorithm applied correctly or efficient computation strategies such as doubling or renaming (eg, 400-8 for 4x98)	**2**
d.	No response or incorrect	**0**
	Correct (6 butterflies) but working and/or explanation indicative of additive thinking (eg, make-all, count all strategy), or incorrect with some indication that the task has been understood in terms of multiplication or division	**1**
	Correct (6 butterflies) with clear explanation in terms of other body parts, eg, "Can't be 7 because not enough feelers."	**2**
e.	No response or incorrect	**0**
	Correct (30 drops) but working and/or explanation indicates an additive approach (eg, counts all, 5+5+5+5+5+5 or uses successive doubling strategy), or incorrect with some indication that the task has been understood in terms of multiplication or division	**1**

	Correct (30 drops) with clear explanation and/or working which indicates an appreciation of proportional relationships (eg, "for each group of 2, zoo needs 5 drops, 6 groups of 2, so 30 drops needed")	**2**
f.	No response or incorrect	**0**
	Correct (22 butterflies) but working and/or explanation indicates an additive approach (eg, counts all, 5+5+5 …), or incorrect with some indication that the task has been understood in terms of multiplication or division	**1**
	Correct (22 butterflies) with clear explanation and/or working which indicates an appreciation of proportional relationships (eg, "5 drops feed 2 butterflies, 55 is 11 times 5, so there must be 2x11 butterflies")	**2**
g.	No response or incorrect	**0**
	Correct (54 butterflies) but working and/or explanation indicates an additive approach (eg, counts all, 5+5+5 …), or incorrect with some indication that the task has been understood in terms of multiplication or division	**1**
	Correct (54 butterflies) with clear explanation and/or working which indicates an appreciation of proportional relationships (eg, see above)	**2**
h.	No response or incorrect	**0**
	Correct (6 butterflies) but no working and/or explanation	**1**
	Correct (6 butterflies), working and/or explanation indicates an additive approach (eg, draws all, counts all, not particularly systematic)	**2**
	Correct (6 butterflies) with clear explanation and/or working which indicates an appreciation of Cartesian product or "for each" idea (eg, tree diagram, systematic list)	**3**
i.	No response or incorrect	**0**
	Correct (18 butterflies) but no working and/or explanation	**1**
	Correct (18 butterflies) but working/explanation indicates an additive approach (eg, draws all, counts all, not particularly systematic)	**2**
	Correct (18 butterflies) with clear explanation and/or working which indicates an appreciation of Cartesian product or "for each" idea (eg, tree diagram, systematic list)	**3**

Example of a Shorter Task:

ADVENTURE CAMP ...

Camp Reefton offers 4 activities. Everyone has a go at each activity early in the week. On Thursday afternoon students can choose the activity that they would like to do again.

The table shows how many students chose each activity at the Year 5 camp and how many chose each activity at the Year 7 camp a week later.

	Rock Wall	Canoeing	Archery	Ropes Course
Year 5	15	18	24	18
Year 7	19	21	38	22

Camp Reefton Thursday Activities

a. What can you say about the choices of Year 5 and Year 7 students?

b. The Camp Director said that canoeing was more popular with the Year 5 students than the Year 7 students. Do you agree with the Director's statement? **Use as much mathematics as you can to support your answer.**

Scoring Rubrics for Adventure Camp:

ADVENTURE CAMP …		
TASK:	**RESPONSE:**	***SCORE***
a.	No response or incorrect or irrelevant statement	**0**
	One or two relatively simple observations based on numbers alone, eg, "Archery was the most popular activity for both Year 5 and Year 7 students", "More Year 7 students liked the rock wall than Year 5 students"	**1**
	At least one observation which recognises the difference in total numbers, eg, "Although more Year 7s actually chose the ropes course than Year 5, there were less Year 5 students, so it is hard to say"	**2**
b.	No response	**0**
	Incorrect (No), argument based on numbers alone, eg, "There were 21 Year 7s and only 18 Year 5s"	**1**
	Correct (Yes), but little/no working or explanation to support conclusion	**2**

	Correct (Yes), working and/or explanation indicates that numbers need to be considered in relation to respective totals, eg, "18 out of 75 is more than 21 out of 100", but no formal use of fractions or percent or further argument to justify conclusion	**3**
	Correct (Yes), working and/or explanation uses comparable fractions or percents to justify conclusion, eg, "For Year 7 it is 21%. For Year 5s, it is 24% because 18/75 = 6/25 = 24/100 = 24%"	**4**

Phase 2

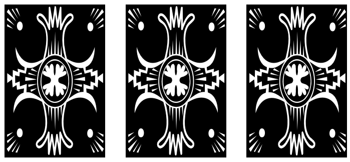
To control order effects and maximise the number of tasks that could be included, four different test booklets were prepared. Each test comprised one of two extended tasks, *Butterfly House* (9 items) or *Tables and Chairs* (13 items), and five shorter tasks (2 to 4 items each). Common tasks were variously used to link the four tests. For instance, two short tasks, *Pizza Party* and *Packing Pots*, were completed by all students, two more, *Adventure Camp* and *Filling the Buses*, were completed by 75% of the students and another two were completed by 50% of the students. The tasks and their respective number of items (shown in brackets) are listed in Table 2 below.

Table 2. Tasks used in Initial Assessment April/May 2004

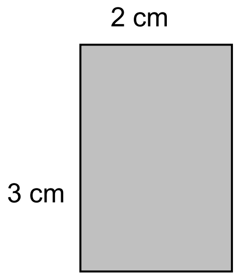
Version 1	Version 2	Version 3	Version 4
Butterfly House (9)	Tables and Chairs (13)	Butterfly House (9)	Tables and Chairs (13)
Packing Pots (4)	Packing Pots (4)	Packing Pots (4)	Packing Pots (4)
Pizza Party (3)	Pizza Party (3)	Pizza Party (3)	Pizza Party (3)
Missing Numbers (2)	Canteen Capers (2)	Adventure Camp (2)	Filling the Buses (2)
Canteen Capers (2)	Adventure Camp (2)	Filling the Buses (2)	Fencing Freeway (4)
Adventure Camp (2)	Filling the Buses (2)	Fencing Freeway (4)	Swimming Sports (2)

Following the initial testing and analyses, and to support the further elaboration of some levels of the *Learning and Assessment Framework for Multiplicative Thinking,* particularly those hypothesised at the upper end of the framework, a number of additional tasks were developed and trialled in October 2004. These were subsequently used to assess all Year 7 students in March 2005. The additional tasks were primarily designed to assess proportional reasoning, the *area* model of multiplication, and students' ability to work with larger whole numbers, fractions, and decimals in more applied contexts involving rates, ratios, and scales. An example of one of these tasks and the associated scoring rubrics is given below.

TILES, TILES, TILES…



Floor and wall tiles come in difference sizes. The basic tile is shown below.



a. How many basic tiles would be needed for an area of 6 cm by 4 cm?

b. How many basic tiles would be needed for an area of 27cm by 18 cm?

c. If the length and width of the basic tile were increased by 2 cm, how many of the larger tiles would be needed to cover 1 square metre (100 cm by 100 cm)?

Show all your working so we can understand your thinking.

TILES, TILES, TILES …		
TASK:	**RESPONSE:**	***SCORE***
a.	No response or incorrect with no working and/or explanation	**0**
	Incorrect (2 tiles), reasoning based on perceived relationship between dimensions eg, “2 goes into 4, 2 times and 3 goes into 6, 2 times”, or correct but little/no working or reasoning	**1**
	Correct (4 tiles), with appropriate diagram and/or explanation	**2**
b.	No response or incorrect with little/no working and/or explanation	**0**
	Incorrect (9 or 18 tiles), reasoning based on factors as above, or correct (81 tiles) but little/no working/explanation	**1**
	Correct (81 tiles), with appropriate diagram and/or evidence of additive strategy, eg, count all or skip count	**2**
	Correct (81 tiles), with appropriate diagram and/or explanation indicating multiplicative reasoning, eg, factors used appropriately	**3**
c.	No response or incorrect with little/no working and/or explanation	**0**
	Some attempt, eg, dimension of larger tile (4cm by 5cm) indicated and /or incomplete solution attempt, eg, attempt to draw all	**1**

	Incorrect, calculation based on incorrect dimension of larger tile, eg, 4cm by 6cm, but supported by correct reasoning of the area required; or correct (500 tiles), with little/no explanation	**2**
	Correct (500 tiles), supported by appropriate diagram and/or explanation based on appropriate diagram or computation strategies	**3**

Concluding Remarks

The primary purpose of this paper was to describe the process involved in developing the tasks and associated scoring rubrics which were used in the *Scaffolding Numeracy in the Middle Years Numeracy Project 2003-2006*. The tasks were administered to a total of just under 3400 Year 4 to 8 students by research and reference school teachers in May 2004 and again in November 2005. Research school teachers administered the tasks and scored these on the basis of the scoring rubrics provided. A professional development session was provided to support this process at a meeting of all research school teachers at the beginning of 2004. Reference school teachers were briefed on the purpose and administration of the tasks at a separate meeting but were not required to score students' work. This was undertaken by a group of scorers under the direction of the research team. A moderation exercise was carried out using a random sample of test booklets from research and reference schools which found that the maximum marker variation was no more than 10%.

The data were analysed using Rasch (1980) measurement techniques, which allowed both students' performances and item difficulties to be measured using the same log-odds unit (the logit), and placed on an interval scale. This process is described more fully in the paper to be presented by John Izard (IZA06376)

The variable maps for each test administration resulting from these analyses were combined to produce an overall list of item thresholds which differentiated items on the basis of student performance. Easier, more accessible items had relatively low item thresholds. For example, the item threshold associated with a score of 1 for part b of the *Tables and Chairs* task (possible scores 0 or 1) was –2.69. The item threshold associated with a score of 4 on part b of the *Adventure Camp* task (possible scores 0, 1, 2, 3 or 4) was 3.53.

A detailed content analysis of items led to the identification of eight relatively discrete categories which described what students might be expected to be able to do if they scored within the corresponding band of item thresholds. The derivation of the categories, which formed the basis of the *Learning and Assessment Framework for Multiplicative Thinking (*LAF) will be described in more detail in the presentation by Dianne Siemon (SIE06377). A summary of this process can be found in Siemon, Breed, Izard and Virgona (2006).

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