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Number Sense and Errors on Mental Computation Tasks

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Abstract

The analysis of errors while completing mental computation tasks reported in this paper represents the first stage of analysis of 5535 test responses from students in Grades 3 to 10 over a period of three years to various subsets of 374 items. Following a previous analysis that suggested the items represented eight increasing levels of difficulty covering nine sub-domains of basic number skills, this report focuses on responses to items at Level 6. Of particular interest are the performances across the grades, the types of errors diagnosed, and the relationship of errors from different types of operations. In analysing errors a developmental approach is adopted, suggesting that more than the "right-wrong" nature of responses is involved. Some errors appear to demonstrate a "partial number sense" that could be used to help construct more complete understanding. Suggestions for future research and classroom practice are made.

Introduction

The debate about the relative importance of various techniques, including mental and written computation, for achieving answers to problems involving basic arithmetic operations has continued for many years in Australia and around the world (see for example, McIntosh, 1990; Reys & Reys, 1986). That important national curriculum documents, for example in the United States (National Council of Teachers of Mathematics [NCTM], 1989, 2000) and Australia (Australian Education Council [AEC], 1991), endorse the development of mental computation as a method of solution, alongside other techniques, has encouraged both research and professional development activity in order to improve students' skills in this area. In Australia, work in Western Australia has involved international test comparisons (Bana, Farrell, & McIntosh, 1997), error analyses of test results (Bana, Farrell, & McIntosh, 1995), and the development of classroom materials (McIntosh, De Nardi, & Swan, 1994), whereas work in Queensland has considered the development of strategies and types of errors particularly for the operations of addition and subtraction based on student interviews (e.g., Heirdsfield, 2002).

Of particular interest for this study is previous research associated with errors that occur when students complete mental computation tasks; however, because of the relatively recent interest in mental computation, most of the research literature has focused on errors associated with written computation. Attention has hence been paid to computational algorithms, often associated with more complex problems than might be expected to be completed mentally (e.g., Ashlock, 1994). In classifying errors identified with written computation, different schemes focus on different levels of detail, from four classes (e.g., wrong operation, obvious computational error, defective algorithm, and random response) to as many as eight or more, with details provided about the particular erroneous procedures (e.g., grouping error, inappropriate inversion, or incomplete algorithm). The types of errors occurring for mental computation tasks may reflect both some of the same mistakes as written computation, for example difficulties with regrouping, and other types of problems, for example a greater likelihood to hear the wrong operation or confuse the order in which numbers are presented.

The research preceding this study reflected two aspects of completing mental computation tasks that are relevant to the work reported here. On one hand, Bana et al. (1995) reported percentages of specific errors for 12 test items completed by students in Grades 3, 5, 7, and 9 in Perth, Western Australia. These errors were often associated with suggested reasons, for example "ignoring place value" or "failure to bridge tens." Bana et al. (1997), on the other hand, reported error patterns from an international study of fraction and decimal *concepts*, rather than actual computation. These test items for example asked for estimation of answers, distinguishing the largest among four fractions, or placing a decimal number approximately on a number line. Although the tasks did not involve computation, they assessed number sense of the type advantageous to students when completing mental computation tasks. This "sense" included judging relative size in terms of "larger" or "smaller," distinguishing "part" from "whole," or appreciating the difference of the "top" (numerator) and "bottom" (denominator) of a fraction.

The examples of number sense exhibited by Bana et al. (1997) fit within the framework proposed by McIntosh, Reys, and Reys (1992). McIntosh et al.'s formal definition of number sense is as follows: "A propensity for and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity (makes sense)" (p. 4). The context of their work was the development of sensible strategies for carrying out mental computation tasks (McIntosh, Reys, & Reys, 1997), obviously with the aim of obtaining correct answers. For the data collected in this study, however, the outcomes observed are responses in a test setting, not in a setting where number sense strategies can be easily discerned by the researchers.

One of the contributions of the current study is the attempt to draw more closely together the interest in the errors students make during mental computation tasks and the aspects of what might be termed "partial number sense" that allow some students to provide answers that although not correct can be seen potentially to be sensible to the student. Teachers are all familiar with two kinds of incorrect answers they observe: those that are close in range to the expected answer and those that are "off the planet" guesses. Part of the aim of this research is to go beyond "right-wrong" and "right (most common and explainable)-wrong" to describe responses that make some number sense in terms of expectation. Such an analysis may lead to a view of students with responses in this category as those with whom special work can be carried out to build upon the partial number sense displayed rather than just place students in a "wrong" basket, which may discourage them from trying to get closer to the exact answer desired. This approach is seen by the authors as being more developmental in nature, acknowledging growth from a complete guess, to an answer that is close to that expected in some sense, to the correct response.

Background

This paper reports on a small part of a larger project whose aims included working with teachers to develop modules for enhancing students' strategy use in performing mental computation tasks, as well as collecting data over three years to document the progression in mental computation skill levels over the Grades 3 to 10. Interviews were also conducted with some students (Caney & Watson, 2003), which helped to provide structure for correct strategies for problems involving rational numbers.

Earlier work by Callingham and McIntosh (2001) demonstrated the effectiveness of Rasch (1980) analysis in providing a model based on the interaction between persons and mental computation items, which resulted in a developmental scale of eight levels. The description of the levels varied from students completing addition bonds to 10 and near 10 at Level 1, to straightforward operations with decimals and fractions at Level 8. This work influenced the larger study from which this report is taken and over three years 5535 tests were completed by students on mental computation items, either taken from the work of Callingham and McIntosh or derived and related to them, to provide a more complete coverage of skills. Altogether 374 items were included in various forms of the mental computation tests over the years and grades. An initial Rasch analysis was completed with data from the first year of the study (Callingham & McIntosh, 2002) and repeated analyses with all of the data confirmed eight levels of mental computation development over the Grades 3 to 10; these are detailed in McIntosh (2004a).

Based on the nine sub-domains of mental computation deemed to be useful for teachers in planning classroom activities, the process of clustering items gave an indication of the relative complexity of tasks across the levels. Table 1 summaries these sub-domains and the levels of the mental computation construct where the items associated with them fell. As examples of the types of tasks likely to be completed successfully at various levels, consider two cases. For Whole Number Single Digit Multiplication and Division, at Level 1 students can double a single digit and multiply a single digit by 10, and at Level 6, they know all table facts and their inverses. For Fractions Addition and Subtraction, at Level 4 students know or can calculate 1/2 + 1/2, whereas by Level 6, they can add 1/2 + 1/3.

Table 1
Sub-domains of Mental Computation with Levels Covered Determined by Rasch Analysis

Sub-Domain	Levels
Whole Number Single Digit Addition and Subtraction	1-3
Whole Number Single Digit Multiplication and Division	1-6
Two Digit Addition and Subtraction	2-7
Two Digit Multiplication and Division	2-8
Decimals Addition and Subtraction	5-7
Decimals Multiplication and Division	6-8
Fractions Addition and Subtraction	4-8
Fractional Operators	5-8
Percents	5-8

The fraction, decimal, and percent items utilised in the tests were meant to be easy enough with the actual numbers involved, to consider the appreciation of the operation on the type of number (for example, fraction or decimal) and number sense in providing a wrong, but not totally unreasonable answer in terms of the numbers involved. For example, for the purposes of mental (not written) computation, fractional operations with common denominators, such as 2/5 + 3/5, were used but complex combinations such as 2/5 + 3/8, were not. It was not deemed to be the purpose of research to test skills very unlikely to be used in daily life.

Because of the large amount of information available for the sub-domains and levels listed in Table 1, the purpose of this paper is to provide a preview of the entire study by considering items appearing only at Level 6 in the mental computation construct. Level 6 is chosen because eight of the nine sub-domains have items at Level 6, providing a variety of tasks and the opportunity to look for links in the errors observed. The specific research addressed here relates to the errors observed for selected items.

1. At Level 6, what are the success and error rates for selected items from each of the eight sub-domains?
2. At Level 6, what are the most common identifiable errors and what percent of errors appear to have an explicable form and in many cases to demonstrate partial number sense?
3. Are there links among the errors observed across different types of items at Level 6?

Methodology

Sample

The sample consisted of 5535 test results from students from schools in the jurisdictions of the Department of Education and the Catholic Education Office in Tasmania, and the Australian Capital Territory Department of Education, Youth and Family Services. Students were in Grades 3 to 10. Data were combined from response sets to all tests over the three years of the project. Overall the numbers of tests administered at each grade level are given in Table 2.

Table 2

Sample Distribution Across Grades

Grade	*n*	Percentage of sample
3	381	6.88
4	342	6.18
5	800	14.45
6	879	15.88
7	940	16.98
8	867	15.66
9	736	13.30
10	590	10.66
Total	5535	100.00

Instrument

Overall 374 items were used across tests, with linked items for all tests to allow the Rasch analysis to be carried out. Tests were presented to pairs of grades: 3/4, 5/6, 7/8, and 9/10. Grade 3/4 students answered fewer items than older students. Of interest in this study are the items that appeared at Level 6 and these are presented in Table 3, along with a brief description of the expectation of the items. For some items, three seconds were allowed for a response and for others, fifteen seconds were allowed. Items are hence annotated short (S) or long (L) in Table 3 but length of time was not considered in the analysis reported in this paper.

Table 3
Level 6 Items in Eight Sub-domains of Mental Computation[1]

Content	Expectations	Examples
Whole Number Single Digit Multiplication and Division	Knows all table facts and their inverses	8 x 7 (S), 7 x 8 (S), 54 ÷ 9 (S), 72 ÷ 8 (S), 56 ÷ 7 (S), 28 ÷ 4 (S), 54 ÷ 6 (S)
Two Digit Addition and Subtraction	Can add and subtract a 2-digit number from a 2-digit number and beyond in special cases	78 + 27 (L), 92 – 34 (L), 105 – 26 (L), 105 – 97 (L), 264 – 99 (L), 62 – 25 (L), 104 – 97 (L), 203 – 197 (L)
Two Digit Multiplication and Division	Can multiply extended basic facts (up to 3-digit numbers by single digit)	40 x 6 (S), 9 x 200 (S), 5 x 40 (S), 60 x 7 (S)
	Can multiply a 2-digit number by 5 or less	14 x 5 (L), 32 x 4 (L), 23 x 4 (L), 24 x 3 (L), 3 x 24 (L), 5 x 24 (L), 17 x 3 (L), 3 x 17 (L)
	Can halve an even 2-digit number	Halve 76 (L)
	Can divide small multiples of 10 and 100 by some small numbers	90 ÷ 5 (L), 200 ÷ 5 (L)
Decimals Addition and Subtraction	Can add and subtract simple decimals (one place only)	0.3 + 0.7 (L), 1.3 + 1.7 (L), 0.6 + 1.4 (L), 0.4 + 0.6 (S), 1.2 + 0.8 (L), 4.5 – 3 (L), 2 – 0.1 (L), 1 – 0.4 (L), 3 – 0.6 (L), 2.2 – 0.3 (L), 5.4 – 3 (L), 4.5 – 3.3 (L)
Decimals Multiplication and Division	Can calculate one half of a decimal (one place)	½ of 0.5 (L), ½ of 0.7 (L)
Fractions Addition and Subtraction	Can add and subtract halves (and equivalents) and quarters beyond one	½ + ¾ (L), ½ + $^{4}/_{8}$ (L), ½ + $^{5}/_{10}$ (L), 1¼ - ½ (L)
	Can quickly subtract a simple unit fraction from one	1 – ⅓(S), 1 – ⅓ (L), 1 – ⅔ (S), 1 – $^{3}/_{10}$ (L)
Fractional Operators	Can calculate a half of a half	½ of ½ (L)
	Can calculate a half or quarter of some 2 or 3-digit numbers	¼ of 120 (L), ½ of 94 (L), ⅓ of 120 (L)
Percent	Can calculate 10% and 25% of some 2-digit numbers	10% of 30 (L), 25% of 80 (S), 25% of 40 (S)

[1] Information extracted from McIntosh (2004a)

Procedure

Teachers administered the mental computation tests in their classrooms using a CD prepared for use on a regular CD-player. The CD was self-contained in providing all instructions for students, who wrote answers only on answer sheets provided.

Analysis

After data from the first year of the project were analysed by Callingham and McIntosh (2002), analysis of the complete data set confirmed the initial construct and the final outcomes for the items in different levels are provided in McIntosh (2004a). An example of items allocated to Level 6 through clustering by the research team is given in Table 3.

The further descriptive analysis reported in this paper was based on aggregated available data for classes who completed items in different years and on different test forms. Sample sizes in the tables in the Results hence vary considerably and there are a limited number of items that reveal the errors made by students in Grade 3/4. This is due to the choice of Level 6 items as the focus for this paper. Descriptive statistics are presented, mainly in terms of percentages, for ease of comparison across different grade levels. Data are initially presented in terms of correct, no attempt, and incorrect responses because the errors are presented in terms of *percent of incorrect responses*, eliminating no attempts. This method is used because it is assumed that no attempt is the equivalent of "don't know" or refusal to guess. What is important in educational terms is the frequency of various types of errors compared with all erroneous answers presented.

Results

The results are presented in eight subsections with examples representing the eight types of operations observed at Level 6. This is followed by a summary of similar features.

Whole Number Single Digit Multiplication and Division

At Level 6, the basic one-digit multiplication and division fact problems are related to products likely to be confusing, such as 6 x 9 = 54 and 7 x 8 = 56, involving factors greater than 5. When presented as multiplication problems, answers are likely to be associated with "close-number" multiplication, for example, when the correct factor is increased or decreased by one (e.g., responding "49" for "7 x 8"). Similarly with division problems, a "near" factor is often given as the answer. Generally the errors conform more to these patterns after Grade 3/4, when students have had more practice with factors between 6 and 9. Related errors for 54 ÷ 9, 54 ÷ 6, and 56 ÷ 7, are summarised across grades in Table 4. In a problem such as 54 ÷ 9 with near factors of 5 and 7, these numbers both represent common errors, whereas for 54 ÷ 6, with near factors of 8 and 10, 10 is an infrequent error, whereas 7 (two off) is more frequent, particularly at Grade 5/6. Perhaps it is at this time when facts are first being learned seriously. Repeating the same factor as given in the statement of the problem also occurs for quite a few students. These errors account for a large percent of incorrect responses but demonstrate an appreciation of the sort of number involved in the operations. The authors believe this is likely to be an exhibition of partial number sense, compared with answers such as "70" or "0.5," which were provided by some students.

Table 4
Summary Information and Comparison of Errors for 54 ÷ 9, 54 ÷ 6, and 56 ÷ 7

	54 ÷ 9				54 ÷ 6				56 ÷ 7			
Grade	3/4	5/6	7/8	9/10	3/4	5/6	7/8	9/10	3/4	5/6	7/8	9/10
n =	298	382	632	474	170	250	101	143	278	380	101	100
% Correct	38.6	49.2	72.3	80.6	20.6	52.4	50.5	82.5	31.7	51.1	42.6	68.0
% No Attempt	40.3	29.8	14.6	13.1	54.1	26.8	20.8	9.8[1]	48.6	30.0	41.6[1]	26.0[1]
% Incorrect	21.1	20.9	13.1	6.3	25.3	20.8	28.7	7.7	19.7	18.9	15.8	6.0
Error												
Factor off by 1	24 (38.1)	31 (38.8)	35 (42.2)	12 (40.0)	10 (23.3)	25 (48.1)	11 (37.9)	6 (54.5)	19 (34.5)	24 (33.3)	5 (31.3)	1 (16.7)
Same factor	6 (9.5)	1 (1.3)	6 (7.2)	1 (3.3)	5 (11.6)	5 (9.6)	3 (10.3)	0 (0)	10 (18.2)	16 (22.2)	1 (6.3)	1 (16.7)

[1] For some 2001 data, errors and no attempts were not distinguishable

Two Digit Addition and Subtraction

Four of the problems at Level 6 involving two-digit whole number subtraction are presented in this section. The first two are 62 – 25 and 92 – 34, for which summary data are presented in Table 5. In this case all grades are sampled but not for both problems.

Table 5
Summary Information and Comparison of Errors for 62 – 25 and 92 – 34

	62 – 25		92 – 34		
Grade	3/4	7/8	5/6	7/8	9/10
n =	271	326	286	776	530
% Correct	10.7	52.2	32.9	58.0	66.8
% No Attempt	52.0	18.1	33.9	11.7	13.2
% Incorrect	37.3	29.8	33.2	30.3	20.0
Error[1]					
Off by 10 – no regrouping [47, 68]	13 (12.9)	3 (3.1)	11 (11.6)	29 (12.3)	16 (15.1)
Off by 10 – over regroup [27, 48]	2 (2.0)	10 (10.3)	2 (2.1)	9 (3.8)	5 (4.7)
Off by 1 [36, 38 or 57, 59]	7 (6.9)	13 (13.4)	8 (8.4)	11 (4.7)	9 (8.5)
Smaller digit from larger	8 (7.9)	19 (19.6)	17 (17.9)	65 (27.7)	17 (16.0)
Within a range of 28-46 or 49-67	53 (52.5)	58 (59.8)	63 (66.3)	130 (55.3)	60 (56.6)

[1] Including some possible explanations

The two problems in Table 5 require regrouping of the tens and units in order to achieve a correct answer. An answer that is off by 10 above (e.g., 47 for 62 – 25 and 68 for 92 – 34) suggests that no regrouping has taken place although the units digit is correct. This is the classic "expected" error. There is some indication, however, that some students may "over-regroup," that is, somehow take an extra 10 away (resulting in 27 or 48). The number of answers only off by one in the units digit suggests a counting on strategy in determining the answer. This tendency was noted by McIntosh (2002) for a wider range of problems but a smaller sample of students in the larger study of which this report is a part. A larger number of errors for each of these two problems, however, is consistent with the error of subtracting the smaller digit from the larger in each case. Why this happens to a greater extent for Grade 7/8 for 92 – 34 is unknown and might profitably be the focus of future research. Finally, if a "sensible" criterion is set as "within 10" of the correct answer, then as seen in Table 5, many students appear to be showing partial number sense. It is of interest to note that few students completed an addition operation rather than subtraction for these problems (3 for each problem).

The other two problems from Level 6 are 104 – 97 and 105 – 97, with summary performances across Grades 5/6 to 9/10 given in Table 6. Overall, for these two problems with numbers bridging 100, the percents correct at each grade were higher than for the previous two problems.

Table 6
Summary Information and Comparison of Errors for 104 – 97 and 105 – 97

	104 – 97			105 – 97	
Grade	5/6	7/8	9/10	7/8	9/10
n =	430	326	149	101	452
% Correct	51.6	68.4	75.2	64.4	70.1
% No Attempt	24.0	15.0	15.4	26.7[1]	15.5
% Incorrect	24.4	16.6	9.4	8.9	14.4
Error					
Off by 1	13 (12.4)	8 (14.8)	1 (7.1)	1 (11.1)	14 (21.5)
Off by 10	2 (1.9)	6 (11.1)	1 (7.1)	0 (0)	4 (6.2)
Subtract smaller number from larger number [3, 13 or 2, 12]	8 (7.6)	2 (3.7)	2 (14.3)	0 (0)	17 (26.2)
All answers between 1-19 (including above)	63 (60)	33 (61.1)	8 (57.1)	3 (33.3)	25 (38.5)

[1] For some 2001 data, errors and no attempts were not distinguishable

The common errors reported in Table 6 were less frequent for these problems than the previous two whole number subtraction problems. It may be expected that counting on and bridging over 100 are mental strategies used successfully by many for these second two problems. The higher success rate and lower percent of "near" errors complement each other. In contrast to the previous problems, Grade 9/10 students appeared to have more difficulty with 105 – 97. It is interesting to note that a few students (7 in total) appeared to have difficulty in completing 104 – 97 with the place value involved, producing answers like 101, 103, 104, 106, and 107, which did *not* make even partial number sense in terms of the numbers in the problem.

Two Digit Multiplication and Division

In this section summaries of student performance on five multiplication problems involving one two-digit whole number are given. For the first two questions, both numbers are odd and they were presented in a different order on different tests: 3 x 17 and 17 x 3. Presented in Table 7 is a summary of the percent of correct, no attempt and incorrect responses for the two items, along with some of the common errors.

Table 7
Summary Information and Comparison of Errors for 3 x 17 and 17 x 3

	3 x 17			17 x 3		
Grade	3/4	5/6	7/8	3/4	5/6	7/8
n =	170	250	101	271	180	326
% Correct	14.1	48.4	49.5	14.0	40.6	65.6
% No Attempt	50.6	26.0	20.8	57.2	27.2	15.0
% Incorrect	35.3	25.6	29.7	28.8	32.2	19.3
Error[1]						
Consider problem as addition [20]	5 (8.3)	2 (3.1)	0 (0)	4 (5.1)	0 (0)	0 (0)
Off by 10: 1 in tens place value [41]	6 (10.0)	11 (17.2)	8 (26.7)	9 (11.5)	16 (27.6)	6 (9.5)
Likely regrouping error [61]	2 (3.3)	2 (3.1)	3 (10.0)	0 (0)	2 (3.4)	0 (0)
Off by 20 – no regrouping [31]	1 (1.7)	4 (6.3)	4 (13.3)	3 (3.8)	4 (6.9)	4 (6.3)
Off by 1 [52/50]	2 (3.3)	3 (4.7)	3 (10.0)	2 (2.6)	2 (3.4)	4 (6.3)
Off by 1 and by 10 [42]	1 (1.7)	3 (4.7)	0 (0)	5 (6.4)	0 (0)	10 (15.9)
Answer ends in "1"	12 (20.0)	21 (32.8)	18 (60.0)	20 (25.6)	27 (46.6)	22 (34.9)

[1] Including some possible explanations

The majority of multiple frequency errors in both problems may reflect procedural difficulties resulting in answers that are not totally unrealistic in a number sense. Misinterpreting the calculation as addition diminishes with grade. Considering the common errors in Table 7, it would appear that there is not very much difference in the difficulties encountered in relation to the order of the multiplication. A mistake likely to be associated with regrouping is much more likely to occur 10 below rather than 10 above the correct answer. The number of incorrect answers ending in "1" is likely to indicate a partial working of 3 x 7 = 21 but difficulties from then on; "321" occurred five times, indicating the classical disregard for the place value involved.

Information associated with performance on the other three problems — 3 x 24, 24 x 3, and 23 x 4 — is given in Table 8. These problems were not necessarily answered by the same students and if so, certainly not close to each other in time during the test. Of interest in particular are similarities when commutivity is involved (3 x 24 and 24 x 3) and when the units digits are swapped (24 x 3 and 23 x 4). The difference in answers (72 for 3 x 24 and 92 for 23 x 4) should be kept in mind in considering responses in Table 8.

Table 8
Summary Information and Comparison of Errors for 3 x 24, 24 x 3, and 23 x 4

	3 x 24				24 x 3				23 x 4			
Grade	3/4	5/6	7/8	9/10	3/4	5/6	7/8	9/10	3/4	5/6	7/8	9/10
n =	271	180	326	149	223	512	776	100	158	286	776	530
% Correct	25.8	57.2	72.7	71.8	33.6	49.6	75.3	67.0	10.1	43.0	60.1	64.5
% No Attempt	46.9	23.9	10.4	15.4	39.1	21.3	8.4	26.0[1]	53.2	31.5	14.7	15.3
% Incorrect	27.3	18.9	16.9	12.8	37.2	29.1	16.4	7.0	36.7	25.5	25.3	20.2
Error[2]												
Answer to other type	3	0	2	0	4	13	11	0	4	14	28	15
[92 or 72]	(4.1)	(0)	(3.6)	(0)	(4.8)	(8.7)	(8.7)	(0)	(6.9)	(19.2)	(14.3)	(14.0)
Consider problem as	3	0	1	0	5	1	1	0	2	0	1	1
addition [27]	(4.1)	(0)	(1.8)	(0)	(6.0)	(0.7)	(0.8)	(0)	(3.4)	(0)	(0.5)	(0.9)
Off by 10 – no	6	7	9	1	4	16	23	1	3	7	8	6
regrouping [62 or 82]	(8.1)	(20.6)	(16.4)	(5.3)	(4.8)	(10.7)	(18.1)	(14.3)	(5.2)	(9.6)	(4.1)	(5.6)
Extra regrouping	3	0	1	2	2	6	2	2	1	2	6	1
[82 or 102]	(4.1)	(0)	(1.8)	(10.5)	(2.4)	(4.0)	(1.6)	(28.6)	(1.7)	(2.7)	(3.1)	(0.9)
Choosing another	1	3	3	3	3	6	2	0	1	1	39	4
multiple not indicated	(1.4)	(8.8)	(5.5)	(15.8)	(3.6)	(4.0)	(1.6)	(0)	(1.7)	(1.4)	(19.9)	(3.7)
[2, 3, or 4]												
Multiply tens but not	2	0	1	0	3	2	0	1	7	3	0	3
units [64 or 83]	(2.7)	(0)	(1.8)	(0)	(3.6)	(1.3)	(0)	(14.3)	(12.1)	(4.1)	(0)	(2.8)

[1] For some 2001 data, errors and no attempts were not distinguishable
[2] Including some possible explanations

In considering the incidence of answers that would be correct for the "other" problem (72 or 92) it should be remembered that it is also possible that the errors are a result of regrouping difficulties by 20. It is more likely for students to make this error with 23 x 4 and this is in fact the most common error for Grades 5/6 and 9/10 for this problem. Apparently misinterpreting the problem as addition is quite rare after Grade 3/4. This may occur because in an oral presentation of problems, "add" and "times" sound very different, whereas in a written format "x" and "+" look similar (cf. Ashlock, 1994). In these problems the regrouping error is probably most likely to be associated with forgetting to regroup 10 at all, resulting in 62 or 82. This is the most common error for all grades less than Grade 9/10 for 3 x 24, for Grades 5/6 and 7/8 for 24 x 3, but not for any grade for 23 x 4. Providing an answer consistent with using a different multiple, from among 2, 3, or 4, occurs for all problems, causing a particular problem for Grade 7/8 students on 23 x 4, with 69 being a frequent response. Answers consistent with multiplying the 10s digit but not the units occur more often with the problem 23 x 4. Overall, the percentage of correct responses improves across grade levels for each item, with a plateau at Grade 9/10 for 3 x 24 and 24 x 3. The number of alternative methods suggested that may be associated with incorrect answers to these problems indicates that many types of partial number sense may be operating.

Decimals Addition and Subtraction

Two addition and three subtraction problems involving decimals from Level 6 are considered in this section. The summary results and errors for the addition items are given in Table 9. In one case the sum bridges to the whole number 1, whereas in the other the answer is 2.

Table 9
Summary Information and Comparison of Errors for 0.3 + 0.7 and 1.2 + 0.8

	0.3 + 0.7			1.2 + 0.8		
Grade	5/6	7/8	9/10	5/6	7/8	9/10
n =	78	776	311	430	427	292
% Correct	34.6	50.6	70.1	56.3	64.2	74.3
% No Attempt	50.0	7.5	12.9	20.7	14.3	11.0
% Incorrect	15.4	41.9	17.0	23.0	21.5	14.7
Error[1]						
No regrouping across place value [0.10 or 1.10]	10 (83.3)	247 (76.0)	39 (73.6)	32 (32.3)	28 (30.4)	7 (16.3)
Inappropriate multiples of digit sum [10, 100, .01]	0 (0)	14 (4.3)	0 (0)	24 (24.2)	22 (23.9)	10 (23.3)
or 20				4 (4.0)	5 (5.4)	5 (11.6)

[1] Including some possible explanations

Grade 7/8 had greater difficulty with 0.3 + 0.7 than 1.2 + 0.8, perhaps suggesting that the presence of a unit digit is a clue to appropriate regrouping. It is interesting that bridging to one and/or choosing an inappropriate power of 10 is quite common in the problem 1.2 + 0.8 and more common than choosing inappropriate powers of ten of the correct answer, as happens for 0.3 + 0.7. Further, partial number sense in terms of an approximate answer "near" to the correct value, does not appear to play a significant role in terms of errors. Difficulties with place value and powers of ten predominate. This may indicate lack of exposure to computations of this type, particularly for the younger students.

The three subtraction problems involving decimals at Level 6 show three differing characteristics. In 4.5 – 3, a whole number is subtracted from a one-place decimal; in 4.5 – 3.3, both numbers are one-place decimals; and in 2 – 0.1, a decimal less than one is subtracted from a whole number. As seen in the summary in Table 10, the percent correct at each grade level drops for the final item.

Table 10
Summary Information and Comparison of Errors for 4.5 – 3, 4.5 – 3.3, and 2 – 0.1

	4.5 – 3			4.5 – 3.3			2 – 0.1		
Grade	5/6	7/8	9/10	5/6	7/8	9/10	5/6	7/8	9/10
n =	172	632	530	180	326	143	78	101	530
% Correct	57.0	61.1	73.0	43.9	61.4	79.0	37.2	40.6	64.9
% No Attempt	7.0	10.9	12.8	37.2	16.9	8.4	52.6	43.6[1]	20.6
% Incorrect	36.0	28.0	14.2	18.9	21.8	12.6	10.3	15.8	14.5
Error									
Place value error for whole number [4.2, .42, or 42]	38 (61.3)	111 (62.7)	36 (48.0)						
[4.47] or [1.09, 1.99, or 1.19]	5 (8.1)	3 (1.7)	0 (0)				1 (12.5)	6 (37.5)	28 (36.4)
Off by 1 in units place	3 (4.8)	5 (2.8)	4 (5.3)	9 (26.5)	16 (22.5)	5 (27.8)	1 (12.5)	0 (0)	5 (6.5)
Off by 1 in tenths place				3 (8.8)	7 (9.9)	2 (11.1)	3 (37.5)	1 (6.3)	2 (2.6)
Place value, other variations [150, 1.05, 15; 1.02, ½, 10.2, .12, 12; 0.9, 0.99, .09, 1.1, .11, .1]	5 (8.1)	5 (2.8)	1 (1.3)	2 (5.9)	7 (9.9)	1 (5.6)	2 (25.0)	3 (18.8)	10 (13.0)

[1] For some 2001 data, errors and no attempts were not distinguishable

As seen in Table 10, the errors made with 4.5 – 3 are consistent with ignoring the fact that 3 is a whole number, not ".3". This may be related to hearing the problem pronounced. In the problem 4.5 – 3.3 there are parallel components that suggest whole and decimal number parts to consider. It is interesting that subtracting 0.1 from a whole number produces more errors of a place-value type in Grade 9/10, perhaps related to some exposure to this type of problem, whereas students in earlier grades may have had more limited exposure and hence not attempted a response. Although producing a wide variety of responses to these questions, students appear aware at least of the *need* to consider the decimal place.

Decimals Multiplication and Division

The only questions involving multiplication or division of decimals that occurred at Level 6 were 1/2 of 0.5 answered by Grade 9/10 and 1/2 of 0.7 answered by Grades 7/8 and 9/10. As seen in Table 11, the former problem was easier, probably because of the association of 0.5 with 1/2 for many students. Both decimal numbers involved an odd digit, which makes the consideration of multiplying by 1/2 or dividing by 2 more interesting. As might be expected the most common errors are associated with the placement of the decimal point in a number involving appropriate digits. This error is more common in the problem involving 0.7. Although these answers are out by a power of 10, in the context of decimal responses they show a consideration of where the decimal point should be placed. These responses appear to show more awareness of decimal issues than the responses that include more than one decimal point in the answer. In Grade 7/8, seven responses (21.2% of errors) answered 0.14 to ½ of 0.7, suggesting a multiplication of 2 x 7 rather than division, combined with a decimal placement difficulty.

Table 11
Summary Information and Comparison of Errors for
½ of 0.5 and ½ of 0.7

	½ of 0.5	½ of 0.7	
Grade	9/10	7/8	9/10
n =	255	101	149
% Correct	75.3	44.6	57.1
% No Attempt	12.2	22.8	21.5
% Incorrect	12.5	32.7	21.5
Error			
0.025 or 0.035	10 (31.3)	1 (3.0)	6 (18.8)
2.5 or 3.5	2 (6.3)	9 (27.3)	5 (15.6)
Multiple decimal points	3 (9.4)	0 (0)	3 (9.4)

Fractions Addition and Subtraction

Two pairs of contrasting fraction problems appearing at Level 6 are considered here. The first pair is 1/2 + 3/4, answered by Grade 5/6 only, and 1 1/4 – 1/2, answered by Grades 5/6 and above, with a summary of performance given in Table 12. With 41.8% of Grade 5/6 obtaining the correct answer to the first question and a similar percent of that grade for the second, it would appear that appreciation of these numbers is similar in the context of either addition or subtraction. The most common error for each problem is to choose the fraction involving 1/2 just above (for addition) or just below (for subtraction) the correct answer; the prevalence of these choices is shown in Table 12.

Table 12
Summary Information and Comparison of Errors for ½ + ¾ and 1¼ – ½

	½ + ¾	1 ¼ – ½		
Grade	5/6	5/6	7/8	9/10
n =	512	382	733	100
% Correct	41.8	47.1	55.8	66.0
% No Attempt	24.2	30.4	24.8	24.0[1]
% Incorrect	34.0	22.5	19.4	10.0
Error				
Revert to nearest half suggested by operation [½ , 1½]	29 (16.7)	18 (20.9)	16 (11.3)	2 (20.0)
Opposite to above	6 (3.4)	7 (8.1)	14 (9.9)	1 (10.0)
Within range of half [1 ↔ 1 ¾, ¼ ↔ 1]	65 (37.4)	46 (53.5)	76 (53.5)	4 (40.0)

[1] For some 2001 data, errors and no attempts were not distinguishable

The answers associated with the “nearest half” occurred often enough to suggest that the operations may have been confused for some. Answers within 1/2 below for the first and less than 1/2 above for the second may indicate a partial number sense for the kind of number that would be appropriate, even if combined with a “guessing” approach. Decimal forms were also mixed with fraction forms of numbers in the incorrect responses but adding denominators and/or numerators was not common. This last observation is in contrast to that of Bana et al. (1995) who found 18 or 19% of Grade 5 students added or subtracted both numerators and denominators in similar problems, also involving halves and quarters.

The other pair of “symmetric” problems at Level 6 was 1 – 1/3 and 1 – 2/3. An overall summary is given in Table 13 along with the common errors. Of the difficulties encountered only four students produced the sum instead of the difference (two for each problem). Difficulties are hence assumed to be of a different nature; for example, near fractions appeared to be distractions. This is an error that persists across grades. The presentation of the number to be subtracted as the answer, may be the result of confusion with the subtraction problem 1 – 1/2, or may just result from a lack of concentration. Two errors occurred individually for the problems, apparently unrelated. Answering “3/4” for 1 – 1/3 may show an appreciation for the size of the number being subtracted but less familiarity with thirds than quarters. Answering 1 1/3 for 1 – 2/3 may indicate an initial recognition of the difference of 1/3 but then forgetting the significance of the 1 and leaving it in the answer. Overall a partial number sense appreciation for the answer being less than one appears to be high, with the percent of answers greater than or equal to 1 given in Table 13.

Table 13
Summary Information and Comparison of Errors for 1 – ⅓ and 1 – ⅔

	1 – ⅓				1 – ⅔			
Grade	3/4	5/6	7/8	9/10	3/4	5/6	7/8	9/10
n =	271	562	202	243	170	250	326	149
% Correct	5.9	34.7	47.5	70.0	26.5	56.4	69.9	63.8
% No Attempt	57.6	33.1	32.2[1]	18.9[1]	51.8	30.4	20.9	22.2
% Incorrect	36.5	32.2	20.3	11.1	21.8	13.2	9.2	14.1
Error								
Near unit fraction [¼, ½]	45 (45.5)	72 (39.8)	14 (34.1)	11 (40.7)	19 (51.4)	14 (42.4)	9 (30.0)	6 (28.6)
Numbers subtracted [⅓ or ⅔]	1 (1.0)	23 (12.7)	10 (24.4)	0 (0)	2 (5.4)	2 (6.1)	5 (16.7)	1 (4.8)
[¾ or 1⅓]	8 (8.1)	18 (9.9)	4 (9.8)	7 (25.9)	1 (2.7)	2 (6.1)	3 (10.0)	3 (14.3)
Answer ≥1	33 (33.3)	36 (19.9)	4 (9.8)	2 (7.4)	12 (32.4)	8 (24.2)	11 (36.7)	7 (33.3)

[1] For some 2001 data, errors and no attempts were not distinguishable

The percents correct reported for each grade for each problem indicate that 1 - 2/3 appears to be easier for Grades 3/4 to 7/8, with closer results in Grade 9/10. Callingham and Watson (in press), who analysed these problems in the wider context of all fraction, decimal, and percent problems used in the study suggest that "counting on from 2" as in 2/3 to 3/3, may be a possible explanation for the better performance of the younger students on 1 – 2/3.

Fractional Operators

Fractions used as operators, as in "1/2 of…," imply multiplication by the fraction or division by the denominator, if the fraction is a unit fraction. For the two problems from Level 6 considered here, the unit fractions 1/4 and 1/3 have denominators that divide evenly into 120. The percents correct for the problems appear similar in Table 14.

Table 14
Summary Information and Comparison of Errors for ¼ of 120 and ⅓ of 120

	¼ of 120		⅓ of 120		
Grade	5/6	9/10	5/6	7/8	9/10
n =	304	78	430	427	292
% Correct	51.6	56.4	43.3	61.6	64.4
% No Attempt	19.7	28.2[1]	31.6	20.8	23.3
% Incorrect	28.6	15.4	25.1	17.6	12.3
Error[2]					
Use "near" fraction	19 (21.8)	4 (33.3)	28 (25.9)	20 (26.7)	8 (22.2)
Mental division error [35 or 45]	8 (9.2)	1 (8.3)	10 (9.3)	3 (4.0)	2 (5.6)
Other values ≤ ½ of 120 [10 – 60]	19 (21.8)	11 (91.7)	38 (35.2)	27 (36.0)	11 (30.6)

[1] For some 2001 data, errors and no attempts were not distinguishable
[2] Including some possible explanations

As seen in Table 14, the most common error at each grade for each problem is apparently to use the other "near fraction" (either 1/3 or 1/4), a similar problem to one identified in the previous section. It also appears to persist across grades. The appearance of answers 5 above the correct answer suggests possibly a division strategy using the denominator but then confusion about whether some part of the whole number is left over for further division. Other responses less than or equal to half of 120 suggest a partial number sense for the type of answer that is reasonable; whether these responses are guesses or approximations is impossible to determine in this setting. Perhaps carelessness is a contributing factor.

Percent

Two percent problems appeared at Level 6: 25% of 80 and 25% of 40, each answered by students in Grades 5/6 to 9/10. No students, however, answered both problems. Summary information is given in Table 15.

Table 15
Summary Information and Comparison of Errors for 25% of 80 and 25% of 40

	25% of 80			25% of 40		
Grade	5/6	7/8	9/10	5/6	7/8	9/10
n =	208	144	333	250	326	149
% Correct	34.1	43.8	79.6	44.4	62.3	67.1
% No Attempt	46.2	34.7	15.3	38.4	19.6	19.5
% Incorrect	19.7	21.5	5.1	17.2	18.1	13.4
Error						
"close" whole number	13	5	7	18	25	11
[10↔35 or 5↔20]	(31.7)	(16.1)	(41.2)	(41.9)	(42.4)	(55.0)
Answer with % symbol	3	5	1	3	5	2
	(7.3)	(16.1)	(5.9)	(7.0)	(8.5)	(10.0)
Number > 80 or 40	2	0	1	1	0	0
	(4.9)	(0)	(5.9)	(2.3)	(0)	(0)

The most common error for each problem appears to indicate an understanding of percent as part of the whole number but a lack of ability to estimate or calculate what that part is. It is interesting to note that there are a low number of responses that are larger than the whole number, indicating some partial sense of the process involved. More students were likely to provide a % sign in the answer, perhaps a sign of lack of number sense.

Links among errors

The Rasch analysis employed in the larger study placed the items considered in this paper along a continuum, which when items were clustered, suggested their difficulty level was sixth out of eight possible categories (Level 8 being the most difficult). Since the items' difficulties are determined by their interaction with the students' responses on a correct/incorrect basis, it is usually of interest to see what is common among the demands of items clustered at Level 6. From a descriptive point of view, however, it is also of interest to ask if there are common features of the errors that occur for these items.

Perhaps the most common link in the potentially explainable errors observed in the tasks explored in this paper is what might be called "closeness." The aspect of closeness varies to some degree with the operation and the numbers involved. In an addition or subtraction problem, a difference of 1 is "close," whereas in a multiplication problem, a difference of 10 may be "close." Choosing a factor 1 off the correct one is "close," as is the unit fraction 1/4 to 1/3, or a decimal off in one place from the correct place value. The "1-off-ness" in some sense, of all of these examples, points to a partial number sense likely to reflect a developmental pathway in student understanding. Almost all items also had some associated answers for which no logical explanation was forthcoming to the authors and that would be placed in a lower developmental category.

In imagining a type of partial number sense that will provide quick feedback on whether an answer is correct, "closeness" seems a reasonable criterion. For operations with the whole numbers, even those involving percent, errors suggest that students are willing to predict answers within a range. The responses for fractions also suggest feasible ranges, such as numbers less than one for a problem of the type "1 – unit fraction." The most difficult problems to fit this closeness criterion are those involving decimals. This suggests that the appreciation of the meaning of decimal numbers is not as well entrenched in the minds of students.

By presenting the results in terms of "no attempts" and "incorrect" it is possible to see trends across grades in how students deal with answers they do not get correct. Often in Grade 3/4 a low correct percent is accompanied by about half of the students not attempting the problem (see Tables 4, 5, and 7). The percent of "no attempts" diminishes after Grade 3/4 but then often stabilises across Grades 7/8 and 9/10.

Discussion

This paper represents the first stage in the analysis of a very large data set in relation to eight levels of mental computation skill. Starting with evidence from a Rasch analysis, responses to items at Level 6 were further analysed descriptively to display the most frequently occurring incorrect responses and in some cases to propose plausible explanations for errors and explore the possibility of links that could explain the difficulties common to the items. The next stage of the research is to consider the clusters of items at other levels and propose other commonalities in the difficulties exposed in student errors. Some further common links may be found but given the differences in items, it would be expected that different characteristics will be identified as well. It will also be interesting to follow the frequency of specific errors across grades for problems at different levels.

The results presented in this paper also lead the authors to suggest that further Rasch analysis may be useful based on a partial credit model (Masters, 1982) rather than a correct/incorrect scoring scheme. In line with other areas of research into students' understanding of mathematical concepts, for example statistical literacy (Watson & Callingham, 2003), developing a hierarchical coding that progresses from no attempts or "wild guesses," to responses displaying partial number sense, to correct responses, may assist in building a more complete model of student understanding, providing more information to educators. Doig and Lindsay (2002) have suggested a similar approach in dealing with errors to multiple choice questions traditionally only marked correct/incorrect and this may be a potentially fruitful area for future research. As an example of what might be hypothesised across the entire spectrum of the mental computation construct, the partial number sense response of failing to regroup the tens digit in two-digit addition (e.g., Table 5) might appear close to items where regrouping is not necessary, such as 34 + 12.

The descriptive analysis of most of the data in this study in three categories of response – correct, incorrect, and no attempt – is a step forward from lumping the incorrect and no attempt responses together. Being able to distinguish the latter two categories not only may be useful to an improved developmental model, but also is likely to be more useful in planning classroom intervention. In most cases in this study the percent of "no attempts" decreased with grade indicating a willingness to engage or "have a go" at the answer. Considering the prevalence of particular errors in terms of all incorrect attempts gives a feel for the seriousness and perhaps the degree of intervention needed.

The issue of addressing the difficulties observed in this study in the classroom is a vexed one. In the first instance the initial focus should be on building strategies for mental computation that will avoid the errors observed here (McIntosh et al., 1997). As part of the larger project from which this study was taken, materials were prepared in consultation with teachers for work with students across the primary grades (Dole, 2004a, 2004b; Dole & McIntosh, 2004; McIntosh, 2004a, 2004b, 2004c). The building of the strategies includes a focus on number sense not related to "closeness" as described here for observed answers but related to the choice of technique for carrying out the calculation. Once errors such as found here are observed in the classroom, it may be helpful to concentrate on the "partial number sense" displayed in order to work for improvement, rather than only to declare answers "wrong."

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