Mental Computation Strategies for Part-Whole Numbers

Annaliese Caney and Jane M. Watson

University of Tasmania

Abstract

Mental strategies used by school students in solving problems involving operations with whole numbers have been documented for some time. There is, however, little information about how students solve fraction, decimal, and percent - or part-whole - number problems mentally. This is despite the fact that there is an increased emphasis on facility with these types of numbers as students move from primary to secondary school. In this study 24 students from grades 3 to 10 participated in individual interviews involving mental computation. This paper will document mental strategies used in operations involving part-whole numbers and relationships. Of particular interest are the links to many of the mental strategies that have been observed with whole numbers. As well, responses will be analysed with respect to their conceptual or instrumental usage. Since many students appear to find part-whole number relationships conceptually challenging, understanding the strategies children use when solving part-whole problems mentally is valuable. This knowledge may help teachers in strengthening students' understanding about intuitive ideas for part-whole numbers, in providing transitions to written algorithms, and in reinforcing estimation practices that will assist in checking answers obtained in other ways, including by calculator.

Introduction

The development of flexible strategies that enable students to use and value mental computation has received considerable attention in many countries. The issues revolve around the skills that are required by students when accuracy is important, when estimation is paramount, and when mental skills are used in conjunction with paper-and-pencil computation and calculators. This interest has been predominantly associated with the four operations on whole numbers, which is not surprising given the early emphasis on whole numbers in mathematics curriculum documents. In Australia, the curriculum profile for mathematics (Australia Education Council (AEC), 1994) devotes one of seven strand organizers within the Number Strand to "Mental Computation." In the six level outcomes described specifically for Mental Computation, the first four are concerned exclusively with whole numbers. At the fifth level, "simple fractions" are mentioned and at level six, the following outcome is expected for students: "Estimates and calculates mentally with whole and fractional numbers, including finding frequently used fractions and percentages of amounts" (AEC, 1994, p. 104).

In New Zealand the emphasis is on facility with whole numbers at lower levels of the curriculum but at Level 3 covering the middle school years, mental methods are one of three approaches to finding "fractions of whole numbers and decimal amounts (including money and measurements)" (Ministry of Education, 1992, p. 41). At Level 5 covering the high school years mental strategies are to be developed for operations with "positive and negative numbers using a calculator, a variety of methods, and other approaches" (p. 49). A similar emphasis on mental computation exists in the latest Standards document released by the National Council of Teachers of Mathematics (NCTM) in the United States. Whereas the focus in the Numbers and Operations Standard of the early grades is on whole numbers, in the standard for grades 6-8, fractions and decimals are mentioned in relation to choosing appropriate calculation methods. This is expanded with a rationale for mental computation and estimation.

> Students should also develop and adapt procedures for mental calculation and computational estimation with fractions, decimals, and integers. Mental computation and estimation are also useful in many calculations involving percents. Because these methods often require flexibility in moving from one representation to another, they are useful in deepening students' understanding of rational numbers and helping them think flexibly about these numbers. (NCTM, 2000, pp. 220-221)

Strategies students use to calculate with whole numbers mentally have been well researched and documented for addition and subtraction (e.g., Heirdsfield, 2001; McIntosh, De Nardi, & Swan, 1994; Threlfall, 2000) and to a lesser degree for multiplication and division (see Heirdsfield, Cooper, Mulligan, & Irons, 1999). An important aspect of this research is that many of the strategies were spontaneous and self-developed by the students. Fostering in students this inventive element in working with numbers is a challenge for mathematics educators in considering the impact of focused teaching strategies for mental computation. The documentation of whole number strategies has created interest in factors that mediate mental computation acquisition and use (Heirdsfield, 2001) and in the transition from mental to written computation (McIntosh, 2002b).

Several factors, besides curriculum suggestions, point to the importance of expanding the focus of mental computation research to include part-whole numbers. First, although much of the research in mental computation has been with early primary children, there is a growing interest in numeracy at the middle school level and a consequent expansion of interest in the types of numbers involved in operations (Caney, 2002). Second, there is considerable evidence that students experience many difficulties in learning and applying rational number concepts including fractions**,** decimals, and percents (Lembke & Reys, 1994; Moss & Case, 1999; Stacey & Steinle, 1998; Steinle & Stacey, 2002; Watson, Campbell, & Collis, 1993; Watson, Collis, & Campbell, 1995). Recently, a report on the findings of the Middle Years Numeracy Research Project (Siemon, 2001) identified "reading, renaming, ordering, interpreting and applying common fractions" and "decimal fractions" as an area with which many students across grades 5 to 9 have difficulty. The introduction of part-whole numbers and relationships challenges students in that they are presented both with the concept of a new number system, and with one that is teeming with multiple representations and connections. In thinking about computational strategies it appears difficult to consider students' mental strategies in operating with part-whole numbers when they often have inadequate understanding of the quantities involved (Moss, 2003).

On a more positive note, it was observed from the results of a large-scale round of mental computation classroom testing that included fractions, decimals, and percents, that students were able to deal effectively with some of the part-whole questions (A. McIntosh, personal communication, 24th May, 2003). Across grades 5-10, for example, the percent of correct responses for the fraction question 1¼ - ½, was 51% (grades 5-6, *n*=413), 58% (grades 7-8, n=484), and 64% (grades 9-10, *n*=140). For the decimal question 0.5 + 0.75, 30% of the grade 7-8 students (*n*=440) answered correctly. In grades 5-6 (*n*=299), 19% of students answered the percent question, 75% of 200, correctly and this increased to 57% of the grade 9-10 students (*n*=314). Although not specifically looking at mental computation strategies, research that looks at difficulties in these areas often asks for explanations that are likely to give clues as to the mental operations students use. Lembke and Reys (1994), for example, report that in solving percent problems in an interview, the repeated halving strategy that uses benchmark percents such as 50%, was the most frequently employed strategy for grades 5 and 7. Hunting, Oppenheimer, Pearn, and Nugent (1998) found that in matching decimal and common fractions a referent of 100 was frequently given in interview explanations, a result similar to observations for mental computation with place value for whole numbers. It appears the links may be expected to be close for whole and part-whole number strategies.

The objective of the current research is to begin documentation of the strategies that students use in mental computation operations involving part-whole numbers: fractions, decimals, and percents. Two perspectives are important in the analysis to be used in the documentation process. Given previous research on mental strategies used with whole numbers it is of interest to see if those strategies will carry over into the realm of fractions, decimals, and percents. Hence, as a starting point, this study uses the comprehensive framework of strategies for single- and double-digit whole number operations appearing in McIntosh et al. (1994, p. 89). This framework is abbreviated and presented in Table 1.

Table 1

Mental Strategies Observed for Whole Number Operations (adapted from McIntosh et al., 1994)

Whole number strategies	
Initial Strategies	Changed division to multiplication Changed subtraction to addition Used commutative law of addition Used commutative law of multiplication
Counting (elementary or larger units)	Counted on/back in ones/twos/tens Counted back to second number in ones/twos/tens Repeated addition/subtraction Used multiples Recited tables
Instrumental place value	Added/removed zeros Used mental form of written algorithm
Relational place value	Added/subtracted parts of a second number Bridged tens/hundreds Used tens/hundreds Worked from the left or right
Used other relational knowledge	Used doubling/halving Used pattern
Used aids	Used fingers Used a mental picture
Other	Knew the answer (i.e., recalled) Guessed

From a second perspective the subsequent use of strategies is also documented and discussed according to Instrumental and Conceptual classifications. This follows the distinction of McIntosh et al. (1994) based on Skemp's (1976) terminology of Instrumental and Relational, although in the current study the term Relational has been replaced by the term Conceptual.

Working instrumentally involves procedural strategies that are learned by rote and have no accompanying explanation that displays conceptual understanding of the processes taking place. Specifically, this study will report on mental part-whole strategy types in terms of their equivalence to whole number mental strategies, categorise observations of strategy use in terms of working instrumentally or conceptually, and give examples from students' explanations.

Method

Subjects

Twenty-four students from grades 3-10 participated in interviews to explain the strategies they used to solve mental computation problems. They were from a primary and a secondary school in the Catholic sector of the Australian state of Tasmania. As part of a wider research project these students were selected for individual interview after completing a paper-and-pencil mental computation class test that included the four operations on whole and part-whole numbers (Callingham & McIntosh, 2002). Three students in each grade (3-10) were selected for interview as representatives of their grades based on three ranges of raw data scores from the class test: 50%-60%, 70%-75%, and greater than 85%. These students were regarded by their teachers as articulate and prepared to discuss their ideas with an interviewer. The initial data set for this study is the set of 151 correct responses provided by these students to questions involving fractions, decimals, and percents.

Protocol

The interview protocol was primarily based on questions taken directly from the class mental computation test items (Callingham & McIntosh, 2002). Some additional items were chosen to reflect the type of problems suggested by the AEC (1994) profile. Not all students were asked the same questions. Examples of the interview questions related to part-whole numbers are presented in Table 2.

Table 2

Example Interview Questions

Decimal	Fraction	Percent
0.75 + 0.5	¾ – ½	25% of 80
4.5 – 3.3	¾ – ¼	10% of 45
3 ÷ 0.5	4 x ¾	90% of 40
0.19 + 0.1	½ of ⅓	150% of 12

Interview Procedure

The individual interviews were conducted with students during class time at their respective schools, half by each author. The interview sessions were approximately 30 minutes and were video taped with parental permission. It was understood by the students that participation in the interviews was voluntary.

Data Analysis

The interview videotapes were transcribed and for the purposes of this report, responses to fraction, decimal, and percent questions were classified using a framework of strategies derived from a framework for solving whole number mental computations (McIntosh et al., 1994). Non-responses (including "I don't know"), uninterpretable responses (including "I just knew" and "I just guessed"), and incorrect responses were excluded from the data set, leaving a pool of 151 responses. Strategy classifications were agreed to by both authors by following a procedure of repeated comparison to possibilities suggested for whole numbers and discussion with each other as suggested by Miles and Huberman (1994, p. 61). Following the grouping of strategies into sets similar to those used for whole number mental computation, part-whole strategies were categorised using the framework of Instrumental and Conceptual understanding (Skemp, 1976, 1986). The results will be presented in two parts. Initially, strategy use derived from parallel observations of whole number operations will be described. In the second part the 151 responses will themselves be classified as Instrumental or Conceptual in relation to the manner in which the strategies were employed. Frequencies will be presented according to the Instrumental and Conceptual classification of strategies observed. To assist in interpreting the language used, Table 3 illustrates the symbols used to translate spoken words. In some cases where departures from these conventions occur, words are written in full.

Table 3

Translation Between Symbols and Words used in Interviews

Symbols	Language
3 ÷ 0.5	"three divided by point five"
4.5 – 3	"four point five take three"
4 x ¾	"four times three quarters"
0.5 + 0.75	"point five add point seven five"
25% of 80	"twenty-five percent of eighty"

Results

Although the sample of students is relatively small, a wide variety of strategies was exhibited for the problems answered. Table 4 summarizes the strategies observed for part-whole numbers and presents examples of each strategy. Many strategies that are employed in solving part-whole number problems appear to overlap with those documented for whole numbers. Some strategies used with whole numbers, for example the elementary counting strategies, were not observed in operations with fractions, decimals, and percents. New strategies that occurred when working with part-whole type numbers have also been added.

Strategies were employed singularly, for example using a memorized rule, or put together as a combination of strategies in a sequential fashion, for example, changing from decimals to fractions, and then bridging to a whole. Twelve problems, including decimals, fractions, and percents, have been selected to demonstrate the different Instrumental or Conceptual processes students display when working with part-whole numbers and to further illustrate the mental part-whole number strategies documented in Table 4. A summary of frequencies of Instrumental and Conceptual responses is presented at the end of this section in Table 5.

Table 4

Observed Part-Whole Strategies

Part-whole strategy	Description	Example
Changed operation	Division to multiplication	For 3 ÷ 0.5, change to multiplication
	Subtraction to addition	For 4.5 – 3, change to addition
Changed representation	Fractions to decimals	For ¾ – ½, change to the decimal problem 0.75 – 0.5
	Decimals to fractions	For 0.5 + 0.75, change to the fraction problem ½ + ¾
	Percent to fractions	For 25% of 80, change 25% to ¼
	Whole number referent of 10/100	For 0.19 + 0.1, 0.19 becomes 19 and 0.1 becomes 10
Used equivalents	Derived from equivalents	For ¾ – ½, ½ is recognised as being equal to 2/4
Used known facts	Derived from a known fact	For 10% of 45, use knowledge of 10% to work out 10% of 40 & 10% of 50
Repeated addition/multiplication	Repeated addition/ multiplication	For 4 x ¾, multiply the ¾ two times and another two times
	Repeated doubling/ halving	In 25% of 80, divide 80 in half and in half again
Used bridging	Bridged to one/whole	For 6.2 +1.9, the 1.9 becomes 2
Worked with parts of a second number	Split by place value	For 10% of 45, divide 40 by 10 and divide 5 by 10
	Split by parts	For 0.5 + 0.75, 0.75 becomes 0.5 and 0.25
Worked from the left/right	Split both numbers separated by a decimal	For 4.5 – 3.3, work from left with whole numbers first or from right with decimal first
	Split by place value after decimal point only	For 0.19 + 0.1, work with the tenths place first
Used a mental picture		For ¾ – ½, divide an imagined picture of rectangle into 4 parts
Used mental form of written algorithm		For 0.5 + 0.75, explain lining the decimals up and carrying
Used memorized rules		For 1.2 x 10, apply the rule "move the decimal to the right"

Percents

All responses to the five percent questions that were used in the interviews, except one, were classified as conceptual. In working with percents students demonstrated facility in employing a repeated halving strategy; for example, a grade 9 student responded to the problem 25% of 80, "I just had all the numbers up to 80 and then I halved 80 and that was 40, and then I halved the 40 and that was 20, that's how I got my answer." This strategy was often used in conjunction with changing percents to fractions, for example, "25% of 80, that's 20. That's a quarter, just like a quarter." For the problem 10% of 45, two different conceptual responses are described. The first involved a response derived from numbers facts in which a grade 7 student derive her answer using her knowledge of 10% of 40 and 10% of 50.

Student:	Um, it's less than 5 ... because it'd have to be 50 to be 5. It's more than 4 so it'd have to be 4.5.
Interviewer:	Why would it have to be 4.5?
Student:	Because, if it's 45, 10% of 40 is 4, 10% of 50 is 5, so it has to be 4.5 because it is in the middle.

When encouraged a grade 9 student explained the same problem, "4.5, divided 40 into 10 … which was 4 and divided 5 into 10 … 0.5." This strategy reflects a similar approach to whole number operations where numbers are split according to place value. This response demonstrated a conceptual understanding both of place value and of the operations involved in finding percents. Alternatively, the response from a grade 9 student, "4.5, just move the decimal place one because, dividing by 10," represents an instrumental, rule-based approach.

Again, the use of number facts featured in the problem 90% of 40 as a starting point from which students derived their answers. For example, a grade 9 student responded "36, 10% is 4 and then you have to times that by 9 to you get 90%" and a grade 10 student, "36, because it's 100% minus 10%, so it's 40 minus 4."

Fractions

For the problem ¾ – ½, responses that were classed as instrumental involved working in a quite complicated way with rules and procedures. A grade 7 student, for example, gave the following response, "One fourth. I had to change the ½ in … it had to have the same denominator and everything, so I changed it to two fourths. Then I took 2 from 3, equals one fourth." A grade 10 student explained a related problem, ¾ – ¼:

Student:	You just have three- no - three fourths take one fourth, so 4 like would be like keep the bottom the same … so you just take the one, that's $^{2}/_{4}$, so that's where I get muddled up with fractions, because the bottom has to stay, if the bottom's the same it's supposed to stay the same. You can't change that... So you take one, two.
Interviewer:	Right, so what would the answer be?
Student:	Two fourths.

In contrast a grade 4 student derived his answer to the problem ¾ – ½ from his conceptual knowledge of equivalents, "¼, because you've got 2 quarters in a ½ and that was ¾ so you take that ½ away which was $^{2}/_{4}$ and there's ¼ left." A different type of conceptual response for the problem involved changing the representation of the fraction to a whole number referent of 100:

Student:	I don't know this sort of stuff … a quarter?

Interviewer: Yes that's right. Why?

Student: Um, because ¾ is, um, I don't know how I worked it out. It's so weird, I just...I had 100 and then a half was like 50, and three quarters was 75, and then I just done 75 minus 50 and that's 25 and that was a quarter. Yes!

Several primary students, including the following grade 3 student, gave quite detailed descriptions of how they had used mental pictures to solve the same fraction problem, ¾ – ½; many hand movements on the table accompanied these explanations.

Student: I just went to the circles, like 1, 2, 3 [points to different parts on desk] and then take ½ and there's a ¼ left.

Interviewer: Right now what did you picture, you put your hands on the table there…

Student: I just like, I pictured my watch [looks at watch] so there's a quarter there, a quarter there, and a quarter there and a quarter there so I just pictured my watch on the table and I just took away ½ and have a ¼ left.

For the problem 4 x ¾, the following response from a grade 10 student was classified as instrumental: "Because that's 4 times ¾ which is twelve quarters, which is 3." A grade 8 response classed as conceptual involved a sequential process that started with a derived number fact and employed a repeated multiplication strategy, "3, well ¾, two ¾ is 1½, so you just times it by 2 and the answer is 3."

Using derived equivalents, for the problem ½ of ⅓, the following response from a grade 8 student was classified as conceptual, "One sixth. If you move the one third, put it up to two sixths, then half of two sixths is one sixth." In comparison, the response classified as instrumental was based on an idea of "doubling" but it appeared to lack a conceptual foundation. The grade 9 student gave the answer, "One sixth" and went on to explain "I don't really know … I knew that I doubled like, if it was like a quarter I doubled the four and then it becomes an eighth, so I just doubled the three, and it became one sixth."

Decimals

In working with decimal problems, many more responses were classified as instrumental compared to responses for the percent and fraction problems (see Table 5). Responses to the problem 0.5 + 0.5 that were classified as conceptual often included the strategy of changing the representation of the decimal numbers to fractions. For example, a grade 4 student reasoned for this question, "1, because if it's 0.5 it's like a ½ and you've got to add the other ½, which is 1." Working instrumentally on this problem involved students referring to the number 10 and using rule-based procedures. An example from a student in grade 6 illustrates these combined strategies: "One whole, like I just, 5 plus 5 equals like 10. Took the zero away." Although this student started with the idea of "a whole," when probed about the zero the student was unable to give any further explanation, stating that it was "just learnt in maths." Similarly in answering 0.5 + 0.75 a grade 7 response classed as instrumental involved the simulation of a written procedure.

Student: That'd be 1.25. Because 0.5 is really point 50, point 5, yeah, point fifty. I just added 50 plus 75 and then put, just moved the decimal, um, would have made 125 so I put the decimal in the middle, after the one [points on desk].

Like the response classed as conceptual for 0.5 + 0.5, the following grade 9 examples of working conceptually with the problem 0.5 + 0.75 involved translating the decimals to fractions, a form of bridging based around the idea of "making a whole," and splitting the numbers into parts.

Student 1: Because 0.75 is like ¾ and then I just added one of the quarters from the half on and that's 1 … and a ¼.

Interviewer: So you added one of the quarters on and then what?

Student 1: And then there was ¼ left, so it's 1¼.

Student 2: 1 and a fourth. Well you have two, seven five, 0.75, you go … there's one whole in that and you've got twenty … twenty-five left, and 25 is a quarter of a whole.

When students work competently from the left (or right) of a whole number problem they often appear to have a conceptual understanding of place value. For part-whole numbers, particularly decimals, however, working from the left (or right) was usually associated with responses classed as instrumental. For example, in explaining 6.2 + 1.9, the following response from a grade 9 student started with the addition of the whole numbers on the left but the response appeared to reflect a written algorithm.

Student: 8.1.

Interviewer: Right. Which did you add first? What were the steps?

Student: I added the 6 and the 1 first and then I added up the 9 and the 2. And then I just, I added the six and the two, hang on, what was the number?

Interviewer: 6.2, 1.9.

Student: I added that and that was 7 and then I added the other two and that was eleven, so I just took the other up to 8.1.

Alternatively a grade 9 response classed as conceptual involved another form of bridging, "Brought it up to, made the 1.9, 2, and then added it on and got 8.2 and took away the 0.1."

In much the same way that students change the operations of whole number problems, for example division to multiplication, a similar strategy was observed in solving the decimal problem 3 ÷ 0.5. The conceptual response of a grade 9 student indicated that he had changed the representation of 0.5 to ½ and the operation from division to multiplication, employing a repeated multiplication or addition strategy, "Um, I just, I knew that 3, I mean I knew that 0.5 was like a half and then I just said that'd be 6 because there's two in one, and there'd be four in two, 6, yeah." In contrast a response classed as instrumental illustrates a student from grade 10 employing a rule for dividing by 0.5, "It's 6. It's 6, because when you divide by 0.5, you double it."

The final question where instrumental and conceptual responses were distinguished involved multiplying 0.25 by 10. Again a rule-based response from a grade 10 student was classified as instrumental, "2.5, move the decimal place to the right one, because there's one zero." A quite different type of conceptual response from a grade 9 student to the same question involved changing the operation from multiplication to division and changing the representation of the decimal to a fraction.

Interviewer: 0.25 x 10. What's the first thing you think about doing?

Student: Trying to divide 10 into 4 ... It's 2.5, yeah, 2.5.

Interviewer: Good. Why were you trying to divide it into four?

Student: Because that's, 0.25 is a quarter.

Table 5 contains a summary of the classifications of responses, Instrumental, Conceptual, and Mixed, for each type of interview question. For most of the basic questions used in this study, there were few responses that were long enough to employ a mixture of conceptual and instrumental strategies. In a few cases, however, it appeared that students combined a rote procedure with a conceptual explanation of some isolated aspect of the procedure. These responses often resulted from students re-explaining their reasoning during conversation with the interviewer. The imbalance of the two types of strategy use for percents and fractions compared to decimals will be considered in the Discussion.

Table 5

Frequencies of Instrumental and Conceptual Interview Responses

	Instrumental	Conceptual	Mixed	Total
Percents	1	22	1	24
Fractions	7	19	2	28
Decimals	58	33	8	99
Total	66	74	11	151

Discussion and Implications

All of the responses where students actively worked out an answer could be associated with a type of strategy use. The responses presented in the Results section are from students who attained correct answers and were selected to clarify application of the strategies used. There were cases, for example of "derived from a known fact" or "worked with parts of a second numbers," which were incorrect. An area for future research is to analyze these types of responses, to compare and contrast strategies and make suggestions for teachers to help repair and strengthen these strategies. It is interesting to note anecdotally that a number of students indicated that they had never been asked in the classroom to explain their thinking for some of the problem types. Like previous research on whole number strategies (e.g., Heirdsfield et al., 1999) it seems that many of the part-whole strategies observed in this study were also self-developed by the students. This may also indicate that recognition is needed by teachers and educators of the importance of extending mental computation to include part-whole numbers: clearly the strategies are available. Competence in being able to compute fraction, decimal, and percent operations mentally may help in facilitating the transition to written algorithms (McIntosh, 2002b) and in strengthening estimation practices.

The fact that some of the whole numbers strategies observed by McIntosh et al. (1994) were not seen to transfer to part-whole numbers in this study, may be an artefact of the problems used in the interviews. Further research with a more inclusive summary of all types of operations involving fractions, decimals, and percents will assist in clarifying the situation with respect to strategies not observed here. A related issue involves the use of percent or fractional splitting as an operator, as in "60% of" or "half of." Although mathematically these might be thought of as equivalents of multiplication because of the use of the word "of," this is not the conceptual equivalent of "lots of" as expressed in 6 x 40 by "6 lots of 40." Watson et al. (1995) found this operator usage to be an important feature of common and decimal fraction understanding and again further research is warranted in this area of mental computation. This last observation may also be related to the significance that multiple representations, for example 25% = ¼, and part-whole numbers equivalents forms, such as $^1/_3$ = $^2/_6$, hold for mental computation. A related issue is students' use and understanding of "benchmarks," those numbers that students appear to encounter and develop facility in using first, for example, ½, 0.5, and 50%, or 0.25, 25%, and ¼. Many of the problems in this study involved numbers considered by some to be "honorary whole numbers" (Hart, 1981, p. 216) in that students develop familiarity and understanding (either instrumental or conceptual) with these numbers but have difficulty with other representations, for example, ⅓ or 35%.

The Instrumental and Conceptual classifications, based on Skemp (1976), provided an avenue to distinguish between those responses grounded in memorized procedures and those responses where students connected their knowledge of part-whole quantities and operations to solve problems mentally. McIntosh (2002a) used similar classifications to investigate common errors in mental computation in grades 3 to 10, and Weber (1999) classified pre- and post-interview responses as procedural or conceptual when exploring the outcomes of a mental computation instructional program for a grade 8 class. It is necessary to note that in the current study the strategies employed by students were not entirely instrumental or conceptual; classifications were made by considering complete responses to a particular problem. In some cases, a response that included a strategy such as using a referent of 10, which would be classified as instrumental, when embedded in a more detailed explanation was classified as conceptual overall. Similarly, there were some responses that appeared to incorporate equal elements of both classifications. Interestingly within the three part-whole domains that were explored during the interviews, responses to the majority of percent problems and fraction problems were classed as conceptual. It seems that understanding based on connections between the different representations is essential in developing mental facility with these types of part-whole numbers. For the decimal problems, however, many more responses were classified as instrumental. Possibly the similarity between decimal and whole numbers, in terms of the look of the numbers and the place value relationships, encourages the use of whole number knowledge in an instrumental way. It is also possible that students have had fewer opportunities in the classroom to build connections between decimals and the other two part-whole representations.

It is envisaged that research on the conceptual understanding from the specific content areas of decimals, fractions, and percents (e.g., Lembke & Reys, 1994) will contribute greatly in further documenting and expanding the appreciation of mental strategies as observed in this study. Another avenue of future research could be associated with the classification of responses based on structural models from cognitive psychology (e.g., Biggs & Collis, 1982; Case, 1985). The single and multiple use of elemental strategies identified in Table 4 appears to be a potentially rewarding starting point.

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