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Experimenting with Quasi-variables in the Teaching of Algebra

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A teaching experiment with two Year 8 classes facilitated early understandings of algebra by focusing on the meaning of equals, expressions, equations, and the variable *x*. Activities highlighted the similarities between arithmetic and algebra by developing generalisations in arithmetic that could be applied in algebra. Knowing that an arrangement of numbers such as in 78 - 49 + 49 = 78 is true because the same number is being added and subtracted has been termed quasi-variable understanding (Fujii & Stephens, 2001). This paper focuses on one component of the teaching experiment, the role of quasi-variable worksheets in the development of variable understanding of changes that leave expressions and equations invariant. In particular, the changes were named the *doing and undoing* of expressions and *doing the same thing to both sides* of equations. Results showed that these quasi-variable ideas were difficult for students to grasp, but there was evidence that they mediated the development of similar variable understandings.

Algebra is an abstract system in which components interact to reflect the structure of arithmetic. Understanding algebraic expressions requires abstract schema (Ohlsson, 1993) of the arithmetic *operations*, the *equals* sign and the *operational laws*, combined with the algebraic notion of *variable*. For example, simplifying by *adding like things* applies in arithmetic for whole numbers (e.g., 5 tens + 2 tens = 7 tens, 50+20 = 70), fractions (e.g., 5 ninths + 2 ninths = 7 ninths, 5/9+2/9 = 7/9), decimals (0.5+0.2 = 0.7) and measures (e.g., 5cm + 2cm = 7cm). It also applies in algebra (e.g., $5x+2x = 7x$). Thus, simplification by *adding like things* is an isomorphic structure underlying both arithmetic and algebra. It is an abstract schema because its meaning lies in terms of the addition process, rather than the particular content (e.g., fractions). The move from arithmetic to algebra requires students' conceptions of operations performed on numbers to be extended so as to develop conceptions of operations on *variables*.

However, bridging the gap between arithmetic and algebra has proven to be difficult, particularly in terms of: (a) abstracting the properties and conventions of operations (Herscovics & Linchevski, 1994); (b) developing meaning for the equals sign (Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Herscovics & Linchevski, 1994); (c) developing meaning for variables (Booth, 1988; Cooper, Boulton-Lewis, Atweh, Pillay, Wilss, & Mutch, 1997; Linchevski & Herscovics, 1996); (d) connecting the knowledge required to solve arithmetical equations by inverting or undoing (backtracking), and the knowledge required to solve algebraic equations by operating on or with the unknown (Booth, 1988; Herscovics & Linchevski, 1994); and (e) overcoming the syntactic similarity between the algebraic notation for $3x$and the arithmetic notation for 2-digit place value (Stacey & MacGregor, 1997).

Carpenter and Levi (2000, p. 1) stated that the "separation of arithmetic from algebra deprives students of powerful schemes for thinking about mathematics in the early grades and makes it more difficult for students to learn in the later grades". Their study found that even grade two children were able to articulate generalisations, particularly those involving zero (e.g., $a+0=a$; $b-0=b$; $c-c=0$), although few in their study were able to generate those of the form $a+b-b=a$. In contrast, Fujii and Stephens (2001) argued that while formal representations such as these may not be appropriate for young children, discussion of the less formal representations associated with numerical examples such as 78-49+49=78 can assist children in developing ideas about underlying algebraic generalisations. This could provide a bridge between arithmetic and algebra, and a "gateway to the concept of a variable" (p. 260), a central concept in the development of algebra. In the preceding example, Fujii and Stephens (2001) used *quasi-variable* to mean the understanding that numbers arranged in this way gave a true sentence regardless of the number that is taken away and then added back. It was the structure of the number sentence that was important because it reflected the more formal algebraic sentence, $a-b+b=a$. In previous papers (e.g., Cooper, Williams, & Baturo, 1999a; Williams & Cooper, 2001), the authors have referred to this idea in terms of an abstract schema (Ohlsson, 1993). In this case, the abstract schema would be *doing and undoing* an expression (leaving the original expression unchanged).

Quasi-variable understanding could be considered to lie behind much of algebra. For example, a common way of solving the linear algebraic equation $3x+6=19$ is to take 6 from both sides, so that the equation becomes $3x+6-6=19-6$. The left hand side now resembles the format above, where the $3x$ has an action +6, which is undone by -6, which results in the left hand side becoming $3x$. Furthermore, the action to take 6 from both sides (and eventually to divide both sides by 3) arises from another generalisation reflecting the formal algebraic sentence, $a*b = c*b$. Here, the equation $a=c$ remains true if the same operation is performed on both sides of the equation. In this case, the abstract schema, or quasi-variable understanding, would be *doing the same thing to both sides* of an equation (which keeps the equation in balance).

The study reported here is a component of a teaching experiment to introduce the notion of variable (e.g., x) by emphasising the common structures of arithmetic and algebra. The experiment took account of Kucheman's (1981) findings that students interpret letters in a number of ways from no meaning, object, specific unknown, generalised number, and variable, with the last three necessary for operations in algebra. It was also based on Usiskin's (1988) argument that the notion of variable be introduced through the approaches: *unknowns* (e.g., solve for x, $3x=6$); generalisations of *patterns* (e.g., 3, 7, 11, 15, ... - the nth term is $4n-1$); and generalisations of*relationships* (e.g., 2® 7, 8® 25, 5® 16 - the relation is $3n+1$; Area of a rectangle is length multiplied by breadth).

This paper focuses on the development of worksheets in arithmetic designed to teach the algebraic abstract schemas *doing and undoing* of expressions and *doing the same thing to both sides* of equations. Quasi-variable understandings of the same processes in arithmetic expressions and equations are developed and extended to variables. The effectiveness of quasi-variables as mediators of variable knowledge is studied. In line with the examples of Carpenter and Levi (2000) and Fujii and Stephens (2001) above, these worksheets aimed at encouraging students to focus on the structure of the expressions and equations and the allowable changes to each, in order to form a foundation for the future manipulation of algebraic expressions and equations.

Method

Sample

The study was undertaken with two classes (53 Year 8 students altogether - 12- and 13-year olds) at a middle-class suburban state secondary school in Brisbane. Twenty 40-minute episodes (separated into two 2-week units, two months apart) to introduce early algebra were taught to one class and repeated with the second. One of the researchers did the teaching whilst the class teacher and another researcher observed. Each teaching episode was videotaped and worksheets used by the students were collected. Fourteen students (seven from each class) were interviewed at the end of the units, their selection based on their responses on the worksheets (ability, attention to homework, other). The monitoring of all student responses and reactions permitted the study of the relationship between teacher actions and student learning. With two classes involved, this meant that modifications arising from the monitoring of one class could be incorporated in successive teaching episodes, particularly those with the other class.

Instructional approach

Instruction followed a *two-path* model (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, & Mutch, 1997) based on the belief that understanding complex algebra is the end product of a learning sequence of mathematical concepts that includes: *binary arithmetic*; *complex arithmetic* (a series of operations on numbers); and *binary algebra*. It means that 2´ 5 and 5+3 (binary operations) are a prerequisite for 2*x* and *x*+3 (binary algebra) while, in turn, 2´ 5-4 and 5+3-4 (complex arithmetic) forms an important prerequisite to understanding 2*x*-4 and 3(*x*-4) (complex algebra). The major purpose of the teaching episodes was to have students reflect on their experiences of arithmetic to draw out their *informal understandings* of arithmetical notions that can be used to underpin formal algebraic knowledge (in line with Kaplan, Yammamoto & Ginsburg, 1989, and Linchevski & Herscovics, 1996). The following sections outline the worksheets and the results relevant to the *doing and undoing* of expressions and *doing the same thing to both sides* of equations.

Worksheets

The worksheets (WS) were used in the teaching experiment to reinforce activity and discussion. The teaching prior to their presentation, which included summarising relevant past work and working examples, provided scaffolding for worksheet completion. In addition, at least the first question on each worksheet was completed for the students. On the numerical worksheets (WS1 to WS4), where the focus was on quasi-variable understanding, initial examples generally used small numbers, then larger numbers on progression through the worksheets to focus the students' attention on the structure of the expression or equation (see Zazkis, 2001). The algebra worksheets were of two types. WS5 and WS6 extended WS3 and WS4 respectively from quasi-variables to variables, but used the same format. WS7 to WS10 began with numerical examples and then moved to variables, each worksheet incorporating further modifications. Not all of the worksheets were worked by each class. This is evident in Table 1 below, and explained in the Results section.

An example from each worksheet is given below in the order in which worksheets were presented to the students, with expected responses provided in curly brackets. For the purposes of this paper, their presentation here has been condensed. In reality the worksheets were enhanced with boxes (WS1), tables (WS3 to WS10), or circled sections (WS9 & WS10).

1. WS1 focused on how what is done to one side of an equation has to be done to the other to maintain equality in the equation.
You are not allowed to calculate anything and you must keep both sides the same.
6´ 8=48 Þ 6´ 8+5=_{6´ 8+5=48+5}

2. WS2 focused on how change has to be undone if the expression is to remain the same.
   Think of some changes that leave the following operations unchanged in value.
   6´ 2 à {6´ 2+9-9; 6´ 2´ 12¸ 3¸ 4}
3. WS3 used a table to reinforce the ideas of WS2 (expressions).
   Action on an expression should not change its value.
   Expression Change Back to original expression Another way
   48+10 (48+10)-6 {(48+10)-6+6} {(48+10)-6+5+1}
4. WS4 used a table to reinforce the ideas of WS1 (equations).
   Equations should be kept "in balance" - both sides must be the same value.
   Equation Operation on one side: Balanced equation
   63¸ 7=7+2 (63¸ 7)+18 {(63¸ 7)+18=7+2+18}
5. WS5 extended WS3 (expressions) to variables.
   Actions on an expression should not change its value.
   Expression Change Back to original expression Another way
   x+10 (*x*+10)¸ 6 {(*x*+10)¸ 6´ 6} {(*x*+10)¸ 6´ 3´ 2}
6. WS6 extended WS4 (equations) to variables.
   Equations should be kept "in balance" - both sides must be the same in value.
   Equation Operation on one side Balanced equation
   p+3=9 (*p*+3)´ 2 {(*p*+3)´ 2=9´ 2}
7. WS7 extended WS5 (expressions), moving from numbers to variables
   Make the right hand column the same value as the left.
   Original expression Same value expression
   (987-489)/7 (987-419)/7+16____ {(987-419)/7+16-16}
   47+2*y* (47+2*y*)/14________ {(47+2*y*)/14´ 14}
8. WS8 extended WS6 (equations), also moving from numbers to variables.
   Keep these the same value on each side!
   Original equation Keeping both sides the same value
   25=526-4008/8 25´ 47=_{(526-4008/8)´ 47}
   57=(6*p*+49) 57/32= {(6*p*+49)/32}
9. WS9 (expressions) moved from numbers to variables, and asked students to circle the original parts of the expressions.
   Expressions stay the same - make the right column the same value as the left
   Original expression Same value expression
   30´ 19+2 (30´ 19+2)¸ 16____ {(30´ 19+2)¸ 16´ 4´ 4}
   (6*p*+49)¸ 32 [(6*p*+49)¸ 32]+17___ {[(6*p*+49)¸ 32]+17-10-7}
10. WS10 (equations) moved from numbers to variables, and asked students to circle the original parts of the equations.
   Equations stay the same on both sides - keep these the same value on each side
   Original equation Keep both sides the same value
   69+9=234/3 (69+9)´ 5´ 17=234/3___{(69+9)´ 5´ 17=234/3´ 5´ 17} 4*y*-19=53
   __________ = 53/8 {(4*y*-19)/8=53/8}

Results

Results for all worksheets (WS1 to WS10) are summarised in Table 1 below. The completion rate refers to the number of questions completed relative to the total number that could be completed, for those students who handed in the particular worksheet. The correctness rate is the number of correct responses relative to the number of questions completed, for those students who handed in the particular worksheet. A discussion of the teaching that preceded the worksheets and the students' responses follows Table 1, and then students' incorrect responses are analysed.

Table 1

Rates for completion and correctness for worksheets, by class

	First Class (n = 26)			Second Class (n = 27)		
Work-sheets	WS handed in (No.)	Completion rate (%)	Correctness rate (%)	WS handed in (No.)	Completion rate (%)	Correctness rate (%)
1	25	70	82	21	51	52
2				20	52	53
3	16	79	72	16	54	66
4	16	55	45	16	47	92
5				20	81	75
6				20	25	91
7	24	83	74			
8	24	96	89			
9				24	84	74
10				23	78	64

Teaching, worksheets, and student responses

WS1 and the first class

The initial teaching episodes for the first class focused on the four operations and equals sign to reinforce their meanings. After this, expressions (e.g., 34+58) and equations (e.g., 4+5=9, 8=10-2; 34+28=31´ 2) were introduced, emphasising first that it is acceptable not to find the answers (to put off closure) and second that some changes can leave expressions and equations invariant (i.e., *doing and undoing* for expressions, and *doing the same to both sides* for equations). WS1 focused on equations because this followed more naturally from the intensive exercises with the meaning of the equals sign.

In general, students from the first class seemed comfortable with not closing expressions and equations. However, after they started work on WS1, it was immediately clear that they were having difficulty seeing what was required in the questions. Half of the questions were then worked by the teacher, who reiterated the need to stop thinking about calculating the answers. As shown in Table 1, there was a 70 percent completion rate on the remaining questions, and 82 percent of the completed questions were correct. However, still the hands went up for assistance, and the students were restless. This suggested that students had difficulty in thinking about equations in this way. Despite the reasonable results, it was decided that a different approach be taken with the second class.

WS1&2 and the second class

Prior to the introduction to expressions and equations, but after the initial episodes relating to operations and equals sign, students in the second class were introduced to arithmetic as transformations. In this approach, operations (e.g., 3´ 2=6) are viewed as change (e.g., 3® [´ 2]® 6). Activities included using calculators to construct a series of changes from a starting number to a finishing number (e.g., 8® [´ 4]® 32® [+50]® 82® [-2]® 80). Students explored the impact of operations (e.g., do they make things bigger or smaller?) and became more aware of what operations (e.g., +20) and inverse operations (e.g., -20) could do. This class was then introduced to expressions and equations in a similar way to the first class, but additional emphasis was placed on the verbal interpretations that could be associated with them (e.g., 5´4: *Five books were sold at $4 each*; 5´ 4=18+2: *Jenny sold 5 books for $4 each. This gave her the same amount of money as Jill, who had $18 in her purse and $2 in her pocket*).

The students appeared more settled than the first class when working on WS1. Fewer hands went up for assistance and the researcher/teacher was required to work only one example from each of the worksheets. However, as Table 1 shows, the completion and correctness rate was much lower than for the first class even though the teacher/researcher had stressed not to work out the answers, just to do the changes; and this continued into WS2. No students in the second class completed all questions correctly for either worksheet. The lower results this time were felt to be partly due to the timing of the worksheets within the lesson, which meant that completion time was reduced (a large number of students did not finish their worksheets), and to the difficulty of the ideas on the worksheets following the relative simplicity of previous ones.

WS3&4 and both classes

Given the above results, these worksheets used a different format, were begun in class, where students could seek assistance, and finished for homework, when they could have more completion time. By this stage, both classes were parallel on the work covered in class, and the teaching immediately prior to the distribution of the worksheets revisited previous work on expressions and equations, particularly reiterating the changes that are allowable. Only 16 students (representing about 60 percent) in each class completed and handed back the homework sheets. Hence, setting the worksheets for homework did not yield the results intended. Some students admitted to having had trouble completing the sheets, so several of those questions were worked with the class. Other students had forgotten to do the homework, or lost the sheets. As Table 1 shows, the completion and correctness rates were higher in the first class for WS3 (expressions); while the correctness rate for WS4 (equations) rate was significantly higher for the second class. Given the high number of unfinished or un-presented worksheets, the figures may not reflect students' abilities; the un-presented worksheets may reflect difficulties.

WS5&6 and the second class

The next teaching episodes developed the notion of variable through Usiskin's (1988) three approaches of unknowns, patterns and relationships, and through materials (e.g., cups and counters). Activities developed in arithmetic were revisited in algebra. For example, Usiskin's (1988) approaches had been introduced within simple and complex arithmetic and then revisited in turn with concrete materials to introduce variable. The teaching of unknowns was based on the transformational approach to arithmetic (see Cooper & Baturo, 1992). The final episodes of the second week of the first unit extended the meaning of variable to more complex expressions [e.g., $3x+2$; $3(x+2)$] and equations (e.g., $3x+2=11$). This extension was also done using Usiskin's approaches and cups and counters. The teaching sequence moved from binary algebra (e.g., $3x$ and $x+3$) to complex algebra [e.g., $3x+2$ and $3(x+2)$]. Throughout, emphasis was placed on the language associated with expressions and equations using variables (e.g., $3(x+5)$: triple the answer to the sum of any number and 5).

The second class had had more exposure to algebra than the first class and was given WS5&6 to reinforce WS3&4. Several examples were worked by the teacher before the students were given WS5 (expressions). WS6 (equations) was only to be completed if time permitted. The students' completion and correctness rates for WS5 were high. There seemed to be a genuine improvement, and six students had all or almost all, answers correct. Furthermore, although the completion rate for WS6 was low, three students finished this worksheet and most of their answers were correct. The switch from arithmetic to algebra did not appear to present too many additional problems for those whose worksheets were available for analysis in the second class.

WS7-10 and both classes

These last worksheets were completed before the lessons moved on to the simplification of expressions and substitution. Again, a different format was experimented with in each class (WS7 & WS8 for the first class; WS9 & WS10 for the second class). By this time, it was observed that in other areas within the teaching experiment, students tended to forget the arithmetic examples when being introduced to variables, and adjustments to the teaching had to incorporate both arithmetic and algebra examples (see Cooper, Williams, & Baturo, 1999b). Hence, WS7 to WS10 began with arithmetic examples before leading into algebra examples. For the first class, the quasi-variable ideas were extended to algebra for the first time in WS7 and WS8. Again, results did not deteriorate with the switch from arithmetic to algebra. In fact, the completion and correctness rates of all worksheets (WS7 to WS10) were high (see Table 1). Many students had all answers correct. To the researchers, it seemed that the students were more comfortable with WS9 and WS10, the last set of worksheets, where they were asked to highlight the original parts of the expression or equation by circling. Perhaps the separation of the original part from the change made it easier to see what was required.

Overall In general, the figures of Table 1 show a steady increase in the completion rates and (with some exceptions) the rate of correct responses. At the end of the teaching, 14 students were asked about the changes that could be done to expressions and equations. Without specific examples being given, only four students remembered the allowable changes for expressions and equations. In general, students counted the above worksheets amongst those that were least liked.

Analysis of incorrect responses (IRs)

Table 2 below summarises the main types of IRs exhibited by students on the worksheets, and the associated percentages. It also displays the percentage of unfinished worksheets. The IRs are then discussed in turn.

Table 2

Percentages of Incorrect Responses on the Worksheets

	% of students showing particular IRs on the worksheets									
Type of IR	WS 1 Equ.	WS 2 Expr.	WS 3 Expr.	WS 4 Equ.	WS 5 Expr.	WS 6 Equ.	WS 7 Expr.	WS 8 Equ.	WS 9 Expr.	WS10 Equ.
1. Incorrect compensation	24	55	41	34	40	15	4	4	4	9
2. Incorrect inverse			22	19					21	
3. Number facts			6	3	10				17	
4. Order of operations			3	6	20			4		
5. Carelessness	4	5	9	9	20		38	21	8	35
Unfinished worksheets-%	43	40	50	53	30	80	29	17	17	22

Initially, the main difficulty that students appeared to have was that they were unsure how to proceed, in terms of correctly compensating for change (IR1 - Incorrect compensation), such as in 74=1332¸ 18 Þ 74-6´ 3={´ 3} (student's response in curly brackets). Sometimes, they misinterpreted what was to be done, for example,*Expression* 125´ 4; *Change* (125´ 4)´ 2; *Back to original expression* {(125´ 2)´ 4}. On WS3 and WS5, where the tasks required students to write the changes to an expression in two ways, students demonstrated they could successfully complete one set of changes but not another, for example, *Expression* p+9; *Back to original expression* {(p+9)´ 16¸ 16}; *Another way* {(p+9)´ 16¸ 24´ 32}. IR1 was less prominent in the later worksheets.

IR2 (Incorrect inverse) occurred when an incorrect operation was used with the correct numbers, for example, *Expression* 19+18; *Change* (19+18)-16*; Back to original expression* (19+18)-16{´ 16}, or a correct inverse operation was used with the wrong numbers, for example, *Expression* 5*m*+6; *Change* (5*m*+6)¸ 3; *Back to original expression* (5*m*+6)¸ 3{´ 1´ 2}. Students with this IR appeared to have problems with equivalence. Often, inverse operations to undo the change were not evident and attention was not always paid to structure, for example, *Original expression* (30´ 19)+2; *Same value expression* [(30´ 19)+2]´ 30{-15¸ 15}. On WS9, this IR was often an isolated mistake, and so might be interpreted as a careless mistake (IR5).

Problems that resulted from IR3 (Number facts) usually interfered with gaining a correct inverse, for example, *Expression* 15-12; *Change* (15-12)¸ 5*; Back to original expression* {(15-12)¸ 5´ 5¸ 0} and *Expression* 9*m*¸ 5; *Change* (9*m*¸ 5)+10*; Back to original expression* {(9*m*¸ 5)+10-1- 8}. These IRs were more frequent with tasks involving expressions.

IR4 (Order of operations) usually meant that brackets were omitted for a correct response, for example, *Original equation* 47+2*y*=3*y*-36*; Keeping both sides the same value* (47+2*y*)/43={3*y*-36/43} and *Expression*: A/3; *Change*: A/3+6; *Back to Original Expression*: A/3+6{-12¸ 2}.

IR5 (Carelessness) was exhibited amongst otherwise correct responses. Usually these errors were unexplainable. However, some occurred on tasks that required students to complete left hand columns when right hand ones were provided, and this was a prominent mistake on WS7 and WS10.

Overall, Table 2 shows a decrease in the number of students who were unsure about the requirements of the tasks in the worksheets, and a decrease in the numbers of unfinished worksheets. Furthermore, in the last few worksheets, IRs were most likely due to carelessness, rather than due to a lack of understanding. Collectively, these patterns suggest that students became more comfortable with the ideas being presented as time progressed.

Summary and Findings

The study reported here was a component of a teaching experiment to introduce the notion of variable by emphasising the common structures of arithmetic and algebra. This paper focused on the role of arithmetic quasi-variable (Fujii & Stephens (2001) activities in the development of variable knowledge. In particular, quasi-variable understandings associated with *doing and undoing* of arithmetic expressions and *doing the same thing to both sides* of arithmetic equations were studied in two Year 8 classes. Worksheets were developed, and discussed in the light of the teaching episodes preceding or accompanying them, as well as the students' completion rates, correctness rates and incorrect responses.

The first finding was that these particular understandings proved to be difficult for the students from these classes to grasp, particularly in the initial stages. This result is open to two interpretations. The first is that it does not support the Carpenter and Levi (2000) position that students can be introduced to quasi-variable generalisations in the early primary grades - the ideas are difficult. However, it must be noted that the quasi-variable understandings sought in this teaching experiment involved more complex arithmetic expressions and equations (Boulton-Lewis et al., 1997) than those studied by Carpenter and Levi (2000). Seeing that (432-78)/13 changed to (432-78)/13+86 must have a compensation of -86 to stay the same value as the original expression is a lot more difficult than seeing that 6+0=6 or that 78-49+49=78. The second interpretation is that the above result does support the early introduction to these quasi-variable ideas, but in a much simpler form. Without prior exposure to these ideas, the students in this study had difficulty with these quasi-variable understandings, given the limited time allocated to this experiment, and the quick progression from smaller to larger numbers and from simple to complex expressions and equations.

The second finding was that the transition from arithmetic to algebra (after WS3 & WS4) appeared to present few additional difficulties for the students, given that completion and correction rates improved after WS3 and WS4. This seems to indicate that the arithmetic quasi-variable activities assisted in the development of understanding of variables for invariance in expressions and equations. However, some caution must be exercised with this conclusion, because the conditions under which some of the worksheets were completed differed (e.g., some were given for homework; some had reduced time for completion), and improvement was no doubt assisted by practice.

The third finding was that if the worksheets are to be used to consolidate generalisations, then attention must be paid to the presentation of the worksheets. The 14 students interviewed counted these amongst the least liked worksheets. To be most effective, it seems essential to have some way of highlighting, for the student, the change being considered and the original expression or equation that is the starting point of the change. In this study, several different formats with boxes, tables and circles were tried. The impression gained by the researchers was that the students were most comfortable with the final worksheets (WS9 & WS10), where the students actually circled the original parts of the expression or equation. This format was visually more appealing, separated the original part from the change, and probably made it easier to see what was required to complete the task. This would be the starting format for worksheets of this type in the future.

The fourth finding was that revision of underlying arithmetic prerequisites such as order of operations, number facts, and inverses is a necessity. This was particularly evident in the IR trends. For example, in the initial worksheets, the main errors were unsureness of how to proceed and an inadequate knowledge of inverse, while in the later worksheets, these errors lessened and the students were then more dependent on their underlying arithmetic knowledge, which in some cases was deficient. Revision is more easily accomplished over an extended period of time by the classroom teacher, rather than within the limited time available to the researchers. However, the initial activities of the teaching experiment that reinforced the meanings (through informal notions) of operations, the equals sign, expressions and equations proved to be successful in helping to provide scaffolding for the introduction of variable and in improving responses for the algebraic worksheets.

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