SIMPLIFYING ALGEBRAIC EXPRESSIONS ®

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Developing ideas about the simplification of algebraic expressions (e.g., 5*x* + 2*x*; 4 x 2*p*; 8*x* + 5*y* + 2*x* - 3*y*) requires the development of abstract schema (Ohlsson, 1993) for multiplication and addition, as well as the algebraic notion of *variable*. For example, simplifying by*adding like things* (e.g., 5 tens + 2 tens = 7 tens, 5/9 + 2/9 = 7/9, 5*x* + 2*x* = 7*x*) is an abstract schema because its meaning lies in the relationships between the numbers and variables, rather than in the numbers and variables. A teaching experiment that aimed to teach simplification procedures through developing arithmetic principles as abstract schema, was conducted with Grade 8 pupils. A variety of activities, including patterns and concrete materials, was employed to highlight the similarities between arithmetic and algebra in the simplification of expressions. This paper describes the activities and their rationale, and discusses the results in terms of the students' responses to the teaching episodes.

Algebra is an abstract system in which components interact to reflect the structure of arithmetic. Understanding algebraic expressions requires abstract schema (Ohlsson, 1993) of the arithmetic *operational laws* and *equals*, combined with the algebraic notion of*variable*. For example, simplifying by *adding like things* applies in arithmetic for whole numbers (e.g., 5 tens + 2 tens = 7 tens, 50 + 20 = 70), fractions (e.g., 5 ninths + 2 ninths = 7 ninths, 5/9 + 2/9 = 7/9), decimals (0.5 + 0.2 = 0.7) and measures (e.g., 5cm + 2cm = 7cm) as well as in algebra (e.g., 5*x* + 2*x* = 7*x*, where *x* is any number, a *variable*). Thus, simplification by *adding like things* is an isomorphic structure underlying both arithmetic and algebra. It is an abstract schema because its meaning lies in terms of the addition process, rather than the particular content (e.g., fractions).

Difficulties in learning algebra have long been documented (e.g., Thorndike et al., 1923), and more recent research (e.g., Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, & Mutch, 1998; Linchevski & Herscovics, 1996) continues to show that achievement rates in algebra are poor. Research indicates that instruction does not seem to be bridging the gap between arithmetic and algebra, particularly in: (a) developing meaning for variables (Booth, 1988; Cooper, Boulton-Lewis, Atweh, Pillay,Wilss, & Mutch, 1997; Linchevski & Herscovics, 1996) and for the equals sign (Behr, Erlwanger, & Nichols, 1980; Herscovics & Linchevski, 1994); (b) connecting the knowledge required to solve arithmetical equations by inverting or undoing (backtracking), and the knowledge required to solve algebraic equations by operating on or with the unknown (Booth, 1988; Herscovics & Linchevski, 1994); (c) overcoming the syntactic similarity between the algebraic notation for 3*x* and the arithmetic notation for 2-digit place value (Stacey & MacGregor, 1997); and (d) abstracting the properties and conventions of operations (Herscovics & Linchevski, 1994). A particular misconception is perceiving variables as letters which begin or are shorthand for an object, such as *a* for apples or *l* for length. Such a misperception has been referred to as "fruit salad algebra" (MacGregor, 1986).

In response to the poor achievement rates, instructional practices which focus on patterns and physical materials to introduce algebra have been developed (e.g., Quinlan, Low, Sawyer, & White, 1993). However, patterns may not be effective as they do not easily lead to the generalisations required for algebraic understanding (Boulton-Lewis et al., 1997; MacGregor & Stacey, 1995); whilst the use of physical materials (e.g., cups to represent variables, counters to represent numbers, balance beam for equals) may impose additional cognitive demands (Halford & Boulton-Lewis, 1992), contain intrinsic restrictions (Behr, Lesh, Post, & Silver, 1983), and have limited connections to symbols (Boulton-Lewis et al, 1998; Hart, 1989).

To improve student learning of algebra, Boulton-Lewis et al. (1997) proposed a *two-path* instructional model (see Figure 1). The model was based on the belief that understanding of complex algebra is the end product of a learning sequence of mathematical concepts that includes: *binary arithmetic*; *complex arithmetic* (a series of operations on numbers); and *binary algebra*. It means that 2 x 5 and 5 + 3 (binary operations) are a prerequisite for 2*x* and *x* + 3 (binary algebra) while, in turn, 2 x 5 - 4 and 5 + 3 - 4 (complex arithmetic) forms an important prerequisite to understanding 2*x* - 4 and *x* + 3 - 4 (complex algebra). It also means that understanding operational laws should be applied to series of operations as well as individual operations, and that learning complex algebra is facilitated by understanding similar (isomorphic) structures in complex arithmetic.

Figure 1. Two-path model (Boulton-Lewis et al., 1997) for algebra instruction.

Sometimes it is difficult to classify activities into the four areas of Figure 1. In this case, instruction to develop algebraic knowledge should be seen as encompassing three stages, namely, arithmetic through pre-algebra (where arithmetic techniques are used with letters, e.g., 3*x* = *x* + *x* + *x*) to algebra (where operations act on variables, e.g., 3*x* + 4*x* = 7*x*) (Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, submitted).

THE STUDY

The study was a teaching experiment (Romberg, 1992) and an intervention design (Hiebert & Wearne, 1991) undertaken with 51 Year 8 students (two classes) at a middle-class suburban state secondary school. Twenty 40-minute episodes (separated into two 2-week units, two months apart) to introduce early algebra were taught to one class and repeated with the second. Simplification of algebraic expressions was part of the second unit. One of the researchers did the teaching whilst the class teacher and another researcher observed. Each teaching episode was videotaped, worksheets used by the students were collected, and a representative sample of 14 students (7 from each class) was interviewed after each unit. The monitoring of student responses and reactions facilitated the modification of successive teaching episodes, and permitted the study of the relationship between teacher actions and student learning.

The major purpose of the teaching episodes was to have students reflect on their experience of arithmetic in order to draw out generalities that can be applied to algebra (e.g., those that differentiate between addition, subtraction, multiplication and division and those that underlie the procedures used in simplifications and equation solving in algebra). The focus of the reflections was informal understandings of mathematical notions that can be used to underpin formal mathematical knowledge (in line with Kaplan, Yammamoto & Ginsburg, 1989, and Linchevski & Herscovics, 1996).

The initial episodes of the first teaching unit focused on the four operations and equals sign with the aim of reinforcing their meanings (e.g., equals means the same value as). After this, the episodes introduced expressions (e.g., 34 + 58) and equations (e.g., 4 + 5 = 9, 34 + 28 = 31 x 2), emphasising the changes that leave expressions and equations invariant (i.e., *doing and undoing* for expressions, and *doing the same to both sides* for equations). The next episodes developed the notion of variable through Usiskin's (1988) three approaches of unknowns (e.g., 3*x* = 6), patterns (e.g., 3, 7, 11, 15, ...) and relationships (e.g., 2Õ 5, 8Õ 23, 5Õ 14), and through materials (e.g., cups and counters). Usiskin's (1988) approaches were introduced within complex arithmetic (see Figure 1) and then revisited in turn with concrete materials to introduce variable. The teaching of unknowns was based on the transformational approach to arithmetic where, for example, 3 x 2 is viewed as a transformation from 3 to 6 through multiplying by 2 (see Cooper & Baturo, 1992). The final episodes of the first unit extended the meaning of variable to more complex expressions [e.g., 3*x* + 2; 3(*x* + 2)] and equations (e.g., 3*x* + 2 = 11). This extension was also done using Usiskin's approaches and cups and counters. The teaching sequence moved from binary algebra (e.g., 3*x* and *x* +3) to complex algebra [e.g., 3*x* + 2 and 3(*x* + 2)].

Teaching episodes in the second unit revised many of the mathematical ideas from the first unit. In particular, further attention was given to reintroducing variable with Usiskin's three approaches (unknowns and transformations, patterns, and relationships), and to modelling simple algebraic expressions and equations with cups and counters. After this, the episodes focused on simplifying algebraic expressions using all four operations. Within these simplification episodes, modelling was used to link informal generalisations expressed in common language to algebra symbols. For example, "the sum of 5 and any other number" was linked to 5 + *n*, and "twice a number subtract 7" was linked to 2*m* - 7). Some modifications were incorporated into the second teaching unit after examination of the first teaching unit (see Cooper, Baturo, & Williams, 1999; Cooper, Williams, & Baturo, 1999). Two findings were particularly relevant to the episodes on simplification of algebraic expressions. The first was to continue to link algebraic expressions to Usiskin's three approaches even after the notion of variable had been revised. The second was to not separate consideration of arithmetic and algebra into different lessons, but to weld together consideration of arithmetic and algebra notions within single lessons.

RESULTS AND DISCUSSION

The episodes on simplification of algebraic expressions were the final teaching in the study and followed teaching episodes on the four operations, the equals sign, arithmetic expressions and equations, the introduction of variable, and the meaning of complex algebraic expressions and equations. In these earlier episodes, the students had difficulty understanding the work on expressions and equations and in understanding the meaning of more complex algebraic expressions and equations [e.g., 3(*x* + 2)], but appeared to understand episodes on equals, unknowns and transformations and simpler algebraic expressions and equations. Patterning and relationships, plus the modelling with cups and counters, appeared to be successful after episodes were simplified and revised. Cooper, Baturo and Williams (1999) and Cooper, Williams and Baturo (1999) contain the results for these earlier episodes.

Teaching episodes for simplification

The teaching episodes and worksheets relating to the simplification of expressions followed the sequence of steps described below. Emphasis was on enabling the students to translate the patterns in arithmetic into abstract schema in algebra. Where appropriate, cups and counters and envelopes and counters were used to model the algebraic schema.

1. M*ultiplication can be represented as repeated addition*. As can be seen in Figure 2, this meaning for multiplication can be translated to algebra. This result paves the way for harder expressions such as 5*r* + 6*s* to be perceived as *r* + *r* + *r* + *r* + *r* + *s* + *s* + *s* + *s* + *s* + *s*.

Figure 2. Multiplication as repeated addition

2. *Expressions of one variable can be simplified by adding/subtracting "like things"*. This step (see Figure 3) was assisted by the ideas in the previous step (e.g., 3 tens + 4 tens = (ten + ten+ ten) + (ten + ten + ten + ten) = 7 tens) and reinforced by modelling with cups and counters (see diagram below). Adding was also extended to subtracting, through inverse notions (e.g., 3*x* + 4*x* = 7*x* means that 7*x* - 4*x* = 3*x*). (Note: It is at this point that difficulties can emerge with "fruit salad" mathematics - the emphasis must be on the variable as "any number".)

Figure 3. Adding "like things"

3. *Expressions can be simplified by multiplying/dividing coefficients.* This step of multiplying coefficients, or the numbers that multiply variables, was developed from transformations as well as by relating arithmetic to algebra (see Figure 4).

Figure 4. Multiplying coefficients

4. *Expressions can be simplified by representing multiplication in terms of area*. As demonstrated in Figure 5, the array or area model for multiplication can be extended to algebra, and then to algebra division by inverse notions (e.g., 3 x 5*a* = 15*a* means that 15*a* / 5*a* = 3; 2*p* x 3*q* = 6*pq* means that 6*pq* /3*q* = 2*p*).

Figure 5. Multiplication as area

5. *Expressions of two variables can be simplified by adding/subtracting "like things".* This was initially achieved vertically with problems ("real situations"), modelled by cups and envelopes (representing different unknowns), then practised for both addition and subtraction examples. Later, these were related to other structures within arithmetic, and eventually horizontal arrangements replaced the vertical ones. This sequence is summarised in Figure 6. (Note: The difference between the algebra and the arithmetic procedures in Figure 6 not only lies in abstractness but also in lack of renaming from one column to the one beside it.)

Figure 5. Multiplication as area

In the algebraic simplification teaching, the episodes from Step 1 (multiplication as repeated addition) had to be repeated due to an incorrect assumption that students understood this multiplication meaning for simple arithmetic. Initially students represented *3x* as three counters and a cup, not as three cups (containing any number). Only after much revision was it evident that students no longer had this misconception (see Cooper, Baturo & Williams, 1999). The episodes from Step 2 appeared to be successful. The understanding of multiplication as repeated addition was strong enough to allow this generalisation. Much effort was expended in later episodes in this Step to ensure that students saw variables such as *a* as "any number" (not as apples), but it was hard to assess whether this was successful.

The introduction of unknown through arithmetic transformations in the first unit was very successful (Cooper, Baturo & Williams, 1999). Thus, the use of transformations appeared to be successful here, and along with the continued use of multiplication as repeated addition (Figure 3) seemed to provide a strong understanding of simplification through multiplication of coefficients. Because of the difficulties in Step 1, the area meaning of multiplication was revised for simple arithmetic at the beginning of Step 4. In the episodes that followed, simplification through multiplication of two variables was introduced. This process of simplification appeared to be understood. However, relating $3x \times 2x = 6x^2$ to $3\text{cm} \times 2\text{cm} = 6\text{cm}^2$ may not provide appropriate understanding for algebraic simplification. In later exercises, this was extended to division with less success than for multiplication.

With the final Step, addition and subtraction of two variables was introduced by relating it to arithmetic and by using two different materials (cups and envelopes) for the two different variables. Problems emerged when different operations were used for different variables. In the arithmetic examples, operations had not been mixed.

Overall, observations indicated that the students received the early lessons (Steps 1 and 2) here exceptionally well, and appeared to understand the generalisations from arithmetic to algebra. However, in subsequent lessons, the problems with subtraction and division became evident. Instruction for simplifying addition and multiplication examples had depended on modelling, but this modelling was difficult to extend to subtraction and division. Instead, instruction relied on extending addition and multiplication work by appealing to the patterns that had already been set up, and on asking students to make common sense transfers (e.g., 3 x 4*a* = 12*a*. Therefore 12a / 3 = 4a). However, due to lack of time, some of the extensions were rushed and appeared not to be grasped by the students. A greater amount of time was needed to be spent on the revision of arithmetic addition and subtraction (e.g., 6 - 13 + 9; $^-6 + ^-8$) prior to the use of negatives in two variable simplification. In addition, more time was needed on two variable multiplication and division operations.

Worksheet and interview responses on simplification

Table 1 summarises the correct responses on the worksheet examples (worked during class time) and the interview examples (administered after the completion of the teaching episodes) for the 14 students selected for post-interview. For the worksheets, the figures represent the percentage of correct responses with respect to the total number of completed examples for each type. For the interview tasks, the figures represent the percentage of correct responses with respect to the total number of tasks for each type given to and worked by the students. The number of interview tasks was considerably fewer than the number of worksheet examples for each type.

Table 1

Percentage of Correct Responses on Class Worksheets and Interview Tasks

Step in teaching sequence	Class Worksheets	Interview Tasks
1. *Multiplication can be represented as repeated addition*	98	43
2. *Simplification for one variable by adding/subtracting like things*.	99	70
3. *Simplification by multiplying/dividing coefficients*	80	100
4. *Simplification by representing multiplication in terms of area*	83	40
5. *Simplification for two variables by adding/subtracting like things*	83	69

The worksheets for Step 1, were successfully completed by the students (98% of responses were correct). By contrast, interview responses relating to interpreting 3*x* as *x* + *x* + *x* were provided by only 6 of the 14 students (43%). However, all 14 students could correctly interpret this expression in the opposite way (i.e., *x* + *x* + *x* as 3*x*). The worksheets for Step 2 were completed very successfully in class (99% correct), but simplifications of this type were not always correct during the interviews. For example, with the simplification of the expression 2*y* - *y*, the interview responses tended to be the incorrect response of 2, which ignored the understanding of 2*y* as *y* + *y*. Students who made this mistake were consistent across all such examples. Hence, the response for simplification of the algebraic expression *x* + *x* + *x* + *k* + *k* was 5*xk*. In the interviews, their responses were similar (e.g., 2*x* + 3*y* + 3*x*+ 5*y* was simplified to 17*xy*). In both the Step 2 worksheet examples and interview tasks, an additional problem was associated with the manipulation of negative values. For example, 3*a* - 7*b* +8*a* - 2*b* was sometimes simplified to 11*a* - 5*b* or 11*a* + 9*b*.

The worksheets for Step 3 were completed reasonably well, but showed a drop in the percentage of correct responses (80%) compared to the earlier worksheets. The only problem here seemed to be associated with the students representing multiplication at too simple a level. Instead of representing 7 lots of 6*q* by 6*q* + 6*q* + 6*q* + 6*q* + 6*q* + 6*q*, students began to represent it by *q* + *q* + *q* +*q* .... and were then reluctant to complete the representation for this large product (42 terms). This did not present a problem in the interviews, when students successfully simplified expressions such as 4 x 2*p* to 8*p.* However, only 8 of the 14 students could represent this result correctly with cups and counters.

The worksheets for Step 4 were also completed reasonably well (83% correct responses). However, responses at the interviews were less than satisfactory. The main difficulty in both the worksheet or interview examples was experienced with division (e.g., 28*pq* / 7*q*; $32x^2$ / 8*x*). The worksheets for Step 5 were as well completed as those for Step 4, and the interview tasks were completed better than those for Step 4. The difficulties occurred with the more complex examples involving negatives.

CONCLUSIONS

For the central notions being developed with respect to addition and multiplication, the teaching episodes described here appeared to be successful in what they were attempting, particularly after earlier ideas were revised. The weaknesses in understanding appeared to lie with the later extensions to subtraction and to division.

Overall, several main conclusions could be made following the results of the study.

1. *Arithmetic as a basis for algebra*. Previous research by the authors (Cooper, Baturo, & Williams, 1999; Cooper, Williams, & Baturo, 1999) had demonstrated that teaching episodes which reflected on arithmetic to build algebra generally worked, but the arithmetic needed to lead straight to the algebra generalisations for each activity. This finding was incorporated in the teaching episodes and worksheets associated with the simplification of algebraic expressions. For algebraic simplification, the link between arithmetic and algebra seemed generally successful. However, generalisations from arithmetic to algebra may be thwarted when understanding of the arithmetic components (e.g, subtraction and division ideas) is missing or defective.
2. *Two-path model*: The two-path model (see Figure 1) appeared to provide a framework for effective teaching. However, rather than consisting of four separate steps to be performed across time, it represents a framework that should be followed for each separate notion and principle and within a single lesson.
3. *Physical materials*: The use of materials (cups, envelopes and counters) acted as a conduit between arithmetic and algebraic notions. In algebraic simplifications, they appeared to work successfully for addition and multiplication, particularly during the classroom work.
4. *Abstract schemas*: In reflecting on this research, it appears useful to consider algebraic principles as abstract schemas. Therefore, the attempt to teach algebra should be seen as an attempt to teach abstract schemas, such as *multiplication as repeated addition* and *adding like things*. The process of simplification has to be extracted away from the particular instances in which they appear. However, the process is arduous for the learner, and, as stated earlier, is easily complicated by missing or defective arithmetic components.

REFERENCES

Behr, M. J., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching*, 92, 13-15.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In Lesh, R & Landau, M. (Eds.),*Acquisition of mathematical concepts and processes*. New York: Academic Press.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra*, K-12. (1988 Yearbook: 8-19). Reston, VA: National Council of Teachers of Mathematics.

Boulton-Lewis, G. M., Cooper, T. J., Atweh, B., Pillay, H., Wilss, L., & Mutch, S. (1997). The transition from arithmetic to algebra: A cognitive perspective. *International Group for the Psychology of Mathematics Education*, *21*(2), 185-192.

Boulton-Lewis, G. M., Cooper, T., Atweh, B., Pillay, H., Wilss, L., & Mutch, S. (1998). Processing load and the use of concrete representations and strategies for solving linear equations. *Journal of Mathematical Behaviour, 16(4),* 379-397.

Boulton-Lewis, G. M., Cooper, T. J., Atweh, B., Pillay, H., & Wilss, L. (submitted). A model of sequential development of knowledge: Arithmetic to pre-algebra to algebra. *Educational Studies in Mathematics*.

Cooper, T. J. & Baturo, A. R. (1992). Algebra in the primary school: Extending arithmetic. In A. R. Baturo & T. J. Cooper (Eds.), *New directions in algebra education*. Brisbane, Qld: Centre for Mathematics and Science Education, QUT.

Cooper, T. J., Boulton-Lewis, G. M., Atweh, B., Pillay, H., Wilss, L., & Mutch, S. (1997). The transition from arithmetic to algebra: Initial understanding of equals, operations and variable. *International Group for the Psychology of Mathematics Education*, *21*(2), 89-96.

Cooper, T. J., Baturo, A. R., & Williams, A. M. (1999, June). *Unknowns, patterns, relationships, concrete materials and teaching the meaning of the algebraic expression 3x*. Paper presented at the South East Asia Conference of Mathematics Education, Manila, The Philippines.

Cooper, T. J., Williams, A. M., & Baturo, A. R. (1999). Equals, expressions, equations, and the meaning of variable: A teaching experiment. In *Making the difference: Proceedings of the 22nd Annual Conference of the Mathematics Education Research Group of Australasia.* Adelaide, South Australia: MERGA.

Halford, G. S., & Boulton-Lewis, G. M. (1992). Value and limitations of analogs in teaching mathematics. In A. Demetriou, A. Efklides, & M. Shayer (Eds.), *Neo-Piagetian Theories of Cognitive Development* (pp. 183-209). London: Routledge.

Hart, K. (1989). There is no connection. In P. Ernest (Ed.), *Mathematics Teaching: The shape of the art*. London: Falmer.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, *27*, 59-78.

Hiebert, J., & Wearne, D. (1991). Methodologies for studying learning to inform teaching.. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 153-176). Albany, NY: SUNY Press.

Kaplan, R. G., Yammamoto, T., & Ginsburg, H. P. (1989). Teaching mathematics concepts. In L.B. Resnick & L.E. Klopfer (Eds.), *Toward the thinking curriculum: Current cognitive research* (pp. 59-81). Reston, VA: Association for Supervision and Curriculum Development.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics*, *30*(1), 39-65.

MacGregor, M. (1986). A fresh look at fruit salad algebra. *The Australian Mathematics Teacher*, 42(3), 9-11.

MacGregor, M., & Stacey, K. (1995). The effect of different approaches to algebra on students' perceptions of functional relationships. *Mathematics Education Research Journal*, *7*(1), 69- 85.

Ohlsson, S. (1993). Abstract schemas. *Educational Psychologist*, *28*(1), 51-66.

Quinlan, C., Low, B., Sawyer, T., & White, P. (1993). *A concrete approach to algebra*. Sydney, NSW: Mathematical Association of New South Wales.

Romberg, T. A. (1992). Perspectives on scholarship and research methods. In D. A. Grouws (Ed.), *Handbook on research on mathematics teaching and learning* (pp. 49-64). New York: MacMillan.

Stacey, K., & McGregor, M. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher, 90(*2), 110-113.

Thorndike, E. L., Cobb, M. V., Orleans, J. S., Symonds, P. M., Wald, E., & Woodyard, E. (1923). *The psychology of algebra*. New York: MacMillan.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford & A. P. Schulte (Eds.), *Ideas of algebra, K-12* (1988 Yearbook: 8-19). Reston, VA: NCTM.