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LEARNING MATHEMATICS: A NEW LOOK AT GENERALISATION AND ABSTRACTION ®

Paul White Michael Mitchelmore

Australian Catholic University Macquarie University

This paper presents a cognitive theoretical framework for the learning of mathematics which has generic implications for other disciplines. The framework has been developed using a combination of established theories about learning and the authors' research into the understanding of some specific types of learning. It is based on the integration of the structure of mathematics as a discipline with the work of Piaget, Skemp, Davidov and others. The key aspect discussed is the role of abstraction and generalisation in both forming mathematical concepts and learning mathematical procedures. Analysis indicates that there are at least two different types of generalisation, the combination of which provides a powerful tool for learning. The paper concludes by analysing some of the authors' recent research in light of the framework, showing how it provides practical guidelines which can be adapted to varying contexts.

Mathematics as a discipline is essentially comprised of definitions, procedures, theorems and proofs. A *definition* is a statement which classifies a set of objects based on some similar generalised properties. Definitions may refer to abstract mathematical objects as in "a rhombus is a quadrilateral with all sides equal". Definitions may also be abstract statements which relate to physical situations. For example, the definition of an angle as "the amount of turning between two lines about a common point" is regarded as encompassing the similarities in all situations involving angles. The angle in the incline of a hill needs to be seen as the amount of turning between the imaginary horizontal and a line which gives the slope of the hill, while the angle in a corner needs to be seen as the imaginary amount of turning between the edges. Mathematical *procedures* are generalisations which describe processes to manipulate certain mathematical objects to achieve desired results. For example, the common way to add fractions requires the fractions being added to have a common denominator. In algebra, only like terms need to be added. *Theorems* are generalisations relating mathematical objects. For example, "the angle sum of a triangle is 180o" relates angles and triangles. Finally, *proofs* are logical arguments establishing the general relations stated in a theorem.

It follows that *generalisation* is fundamental to mathematics as a discipline. How important is it in the way mathematics is learnt?

GENERALISATION AND ABSTRACTION

One brief of school mathematics curricula is to introduce school students to a hierarchal mathematical structure consisting of an increasingly complex array of abstract mathematical concepts. As students progress through school, the mathematics they learn becomes more formalised. This progression in concept development can be summarised as follows:

Everyday concepts Æ Elementary mathematical concepts Æ Formal mathematical concepts

Dog, door and car are examples of everyday concepts. The concept of numbers such as two and shape recognition based on how the shape looks (rather than on the shape's properties) are examples of elementary mathematical concepts. The concept of rational number and shape recognition based on the shape's properties are examples of formal mathematical concepts. The aim of school mathematics is to build elementary concepts, based on analysis of everyday objects, events, situations or concepts, which can be later appropriately modelled by the "approved" formal concepts and their associated definitions. The manner, then, in which such abstract mathematical concepts are formed is central to effective mathematics learning as is an understanding the nature of "abstraction".

Abstraction as a Process

A common theme in the work of Dienes, Piaget and Skemp earlier this century is that concepts are formed when experiences are collected together on the basis of their similarities (Mitchelmore & White, 1995). Generalising leads to the class of experiences being seen in a new way so that a new experience (which may look quite different to the original ones) can be classified as belonging or not belonging to that class. The key ideas underlying this approach to concept formation are classification, similarity, abstraction and concept. The relation between these ideas has been concisely summarised by Skemp (1986):

> *Abstracting* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An*abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept.* (p. 21)

In the case of everyday concepts, the similarities which link objects in everyday classes frequently relate to the objects' purpose and are not usually definable in terms of single attributes. "Real-world attributes, unlike the sets often presented to laboratory subjects, do not occur independently of one another" (Rosch, 1977, p. 213). Everyday objects are classified on the basis of a cluster of attributes. For example, an object may be classified as a spoon on the basis of its function and shape. At some point, classes of objects begin to function as single entities. Children who play the "scissors, paper, rock" game with their hands are operating with classes and not with individual objects. Words such as *scissors* now denote concepts-new mental objects, created out of classes of concrete objects, which can be related to each other without reference to specific, concrete objects (Greeno, 1983).

Elementary mathematical concepts are formed through the same process of classification and abstraction as everyday concepts, but with one important difference: the objects which are classified are not concrete objects nor even mental objects (everyday concepts), but relations between everyday objects or concepts. "The [mathematical] abstraction is drawn not from the object that is acted upon, but from the action itself" (Piaget, 1970. p. 16). More recently, Sfard (1991) has argued in a similar vein that a great number of elementary mathematical concepts are reifications of mathematical operations. She uses this term to refer specifically to the conversion of a mathematical process into an object of thought. The most obvious example is that of a whole number (such as two), which can be seen as developing from the action of counting or, in Sfard's terms, the reification of the counting process.

As with everyday concepts, higher-level mathematical concepts may be formed by abstraction from existing concepts. For example, the concept of *number* may be formed by abstraction from the concepts and actions on or relationships between whole number, integer, rational number, real number and complex number. A difference between mathematical and everyday concepts is the increasing use of definitions. For example, young children may only be taught to recognise a circle visually (elementary mathematical concept), but by secondary school they are expected to be able to define it in terms of centre and radius (its properties). There is, then, a clear link between concepts (in learning) and definitions (in mathematics as a discipline). Students' ability to operate with definitions must necessarily depend on how well the concepts associated with a definition have been developed. Vinner (1991) have described this interplay as a potential conflict between *concept-image* and *concept-definition*.

In summary, "abstraction" is usually taken to denote a constructive process-not the product of the process-which proceeds in a cycle of three stages:

• the actions of recognising similarities and ignoring differences,

• familiarisation with the concept which embodies these similarities,

• reification of the concept into an object in its own right.

Abstraction as a Product

A cursory look at much mathematics teaching would suggest that it does not focus on process, but the opposite-abstraction as a product. The result of such a focus is that many students come to believe that mathematics consists of a large set of rules with little or no connection to each other and hardly any relevance to their everyday lives (Graeber, 1993; Schoenfeld, 1989). Mitchelmore (in press) describes this product orientation as the ABC method-teaching **A**bstract **B**efore **C**oncrete-and explains that it is widely used in mathematics teaching. The theory is that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). Steinbring (1989) shows that many school mathematics teachers believe procedures should be learnt before applications, and textbooks and even curriculum guides often reflect the same order.

The belief in the abstract Æ concrete learning order would seem to be based on two assumptions. The first is a Platonic philosophy, in which mathematics is viewed as a perfect system that is hidden from sight because of the imperfections of the real world but can nevertheless be revealed by good teaching (Ernest, 1991). The second is a functionalist psychology approach, where "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p.9). Neither of these assumptions has any room for abstraction as a process, and both lead to a belief in teaching as the transmission of knowledge from one who knows to one who does not (Thompson, 1984).

In summary, educational practice would often seem to be based on the following notions of abstraction.

• Abstract mathematics may be learnt in almost complete separation from the world of experience.

• Abstractions can be taught directly by careful choice of examples.

• Abstractions should be learnt before their applications.

The Two Faces of Abstraction

The process versus product views of abstraction presented shows clearly that the word "abstract" does have different meanings to different people. The two meanings which predominate are the ideas of generality and apartness which have been described respectively as *abstract-general* (process) and *abstract-apart* (product) (Mitchelmore & White, 1995; White & Mitchelmore, 1996). We have argued that mathematical concepts and procedures learnt in an abstract-apart manner are limited because they can only be applied in situations which look suitably similar to the context-free way in which they were learnt. Given that concepts and procedures have been shown as key aspects of mathematics, we now consider an example for each.

The concept of angle and noting that the definition of angle uses amount of turning between two lines joined at a point, an abstract-general concept of angle would allow the lines and turning to be identified in any angle situation. On the other hand, learning about angles from abstract diagrams with the two lines visible risks the development of an abstract-apart concept where angles can only be identified in situations which look like the diagrams. The existence of such an abstract-apart concept is demonstrated by Mitchelmore and White (1998) who showed that in situations when two lines are visible (eg: corners), students as young as grade 2 could identify the angle, whereas when no lines are visible (eg: rotations) nearly 40% of grade 8 students could not identify the angle (in particular, the two lines).

A procedure for division with indices is the "rule" *am* ~ *an* = *am-n*. A common abstract before concrete method for teaching this rule is via . The result is that students will often correctly simplify *a6* ~ *a4*as *a2*, but when confronted with the different looking 56 ~ 54 will reply with 12 (even though the "*am* rule" should be seen as a generalisation of the 56 number situations).

Speaking of students who have been taught by the ABC method, Dreyfus (1991, pp. 28) writes: "They have been taught the products of the activity of scores of mathematicians in their final form, but they have not gained insight into the processes that have led mathematicians to create these products." Students have often been taught by definitions, but as Skemp (1986, p. 25) puts it, "concepts of a higher order than those which people already have cannot be communicated to them by a definition".

Generalisation in the Process of Abstraction

One process approach diametrically opposed to the ABC method is the "concrete to abstract" approach where students are encouraged to model the pattern-seeking behaviour which is said to be fundamental to mathematics using concrete (manipulative or figural) materials.

One manifestation focusing on a theorem may look like the following example:

> Grade 8 students are presented with a whole range of drawn right angled triangles and asked to measure the sides, square them and look for a pattern. Answers show that squaring the longest side (the hypotenuse) is usually about equal to adding the squares of the other two sides (Pythagoras' Theorem)

Activities like these do show how new mathematical knowledge can be generalised, but the results are still mysterious. Some obvious questions are simply not asked: "*Why* does Pythagoras theorem hold?" "Are you *sure* the relationship always holds? Exactly?" If these questions are not asked, there is a constant danger of making false inductions. More importantly, it is answering the "Why?" questions which explains the results, and this is surely the essence of mathematics.

Davidov (1972/1990) would describe the pattern-seeking behaviour activity above as *empirical generalisation*. He ascribes the pedagogical principle, "always proceed from the particular to the general," to a wide-spread belief in the psychological principle of empirical generalisation as the basis for all learning and teaching in school (not only mathematics). At the same time, Davidov criticises empirical generalisation on several grounds:

> • Classes [generalisations] are not formed by noticing common features alone-there is always some reason for objects being grouped together (often a common function). Classifying on the basis of external characteristics does nothing towards identifying their inner connections.
>
> • Teaching through empirical generalisation must consist of the transmission of concepts [procedures/ theorems] known to the teacher (who is aware of the inner connections) through examples chosen by the teacher but which to the students appear to be unrelated.
>
> • A restriction to empirical generalisation, by emphasising the link to concrete experience, discourages children from embarking on something which is qualitatively different: abstract mathematical thought.

Returning to the Pythagoras example, Davidov's criticisms match our own concerns. In particular, the activity, not in any way showing *why* Pythagoras' Theorem holds, exemplifies the fact that classifying on the basis of external characteristics does not identify inner connections. Hence, to be effective, the superficial similarities delivered by empirical generalisations need to be supplemented (integrated) with considerations of deeper, structural similarities which do identify the inner connections. Such connections occur in what Davidov calls *theoretical* or *content-based generalisation.* A few quotations should clarify the new way in which this theory uses the same terms as in empirical generalisation, plus some new ones:

> Generalisation is achieved, not through simple comparison of the attributes in particular objects ... but through analysing the essence of the objects and phenomena being studied. (p. 295)
>
> To make such a generalisation means to discover a principle, a necessary connection of the individual phenomena within a certain whole, the law for the formation of that whole. (p. 295)
>
> Only when the origin of the object or a conception is clear to the student does it become possible to assert that ... the student has a concept of that object. (p. 334)

We may note some particular aspects of these ideas:

> • The emphasis on analysis, in place of comparison or similarity, as the means for identifying the concept to be abstracted. The term *content-based* emphasises that analysis can only operate on content, not on superficial appearances.

- The emphasis on connections (not only between elements but also between the general and the particular) which exist for some reason, are necessary, the result of some law. It is analysis which brings out these necessary connections.

The following alternate approach to Pythagoras' Theorem is presented as embracing analysis and theoretical generalisation.

Consider four copies of the one right angled triangle formed to make a square as shown in Figure 1. The inside is also a square. The triangles are then rearranged as shown in Figure 2.

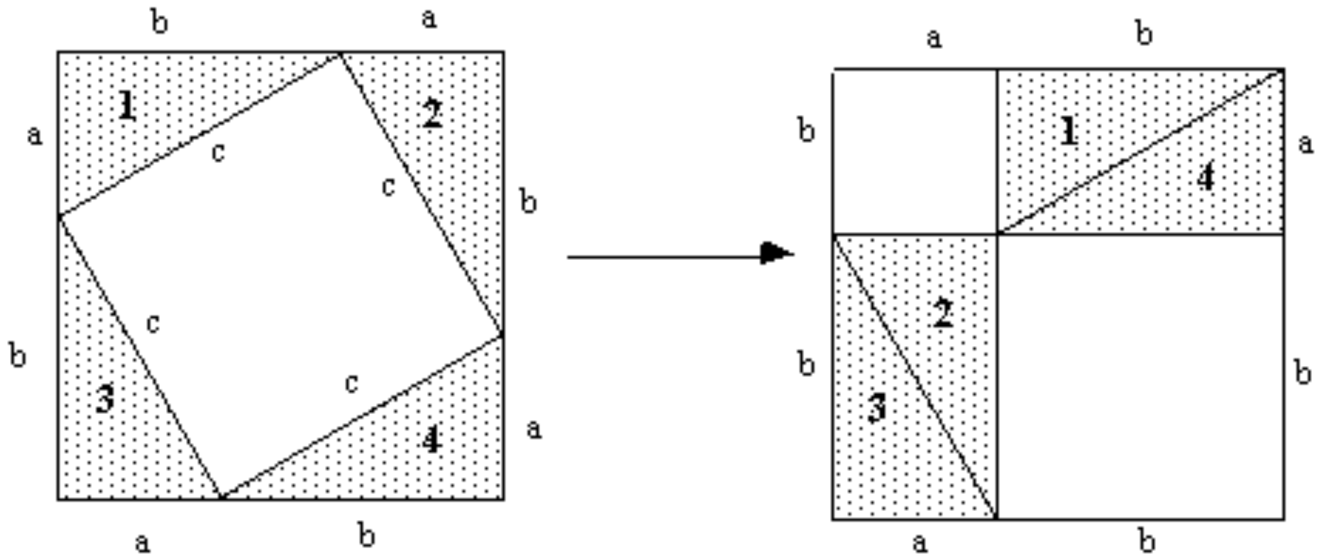

Figure 1: Triangles' first configuration Figure 2: Triangles' second configuration

Since the total area is unchanged in the rearrangement, the uncoloured area in both figures is unchanged and so Pythagoras' Theorem is established. This demonstration uses particular triangles, but the process is in no way dependent on particular cases nor approximations, and so establishes the general result. The activity also shows why the theorem holds. In particular, the squaring aspect of the theorem is now linked to the geometrical figure of a square.

A Theoretical Framework

The analysis above provides a more in depth understanding about the nature of abstraction and generalisation in mathematics. This analysis is summarised in Table 1.

Table 1

A summary of the nature of abstraction and generalisation in learning.

Abstraction	Generalisation
<u>abstract-apart</u> A product based approach where learning occurs in almost complete separation from	<u>empirical generalisation</u> A pattern-seeking behaviour where a generalisation is formed inductively from a

the world of experience. abstract-general A constructive process involving the recognition of similarities and ignoring of differences.	range of examples without examination of why the generalisation holds. theoretical generalisation The generalisation is formed by considering deeper, structural similarities which identify inner connections.

We note that the terms generalisation and abstraction are often used interchangeably in the literature. The essential difference, as we see it, is that abstraction creates a new mental object (a concept) whereas generalisation extends the meaning of an existing concept. The act of abstracting is based on generalising, but is seen as qualitatively different from simply identifying patterns in a set of examples. It is a many to one function where generalisations are synthesised from many inputs to form a new abstraction. Dreyfus (1991) summarises the process as a sequence of generalising Æ synthesising Æ abstracting.

Integrating the ideas of abstract-general concepts, empirical generalisation and theoretical generalisation provides a cognitive framework for facilitating learning in mathematics (and we claim other disciplines) in line with constructivist theories. The basis of the framework is generalisation-which is entirely sensible given the nature of mathematics as a discipline. The abstract-general process of recognising similarities and ignoring differences, familiarisation with the concept which embodies these similarities and finally reification of the concept into an object in its own right is promoted via generalisation. We, thus, argue for the use of examples or concrete referents as a starting point for developing concepts and learning mathematical procedures. However, even though the role of empirical generalisation is seen as important (necessary), it is not sufficient and some form of theoretical generalisation is seen as vital to mathematical process learning and forming abstract-general concepts. Theoretical generalisation also provides initial thinking patterns conducive to introducing formal mathematical proof. Figure 3 presents a diagrammatic version of the theoretical framework.

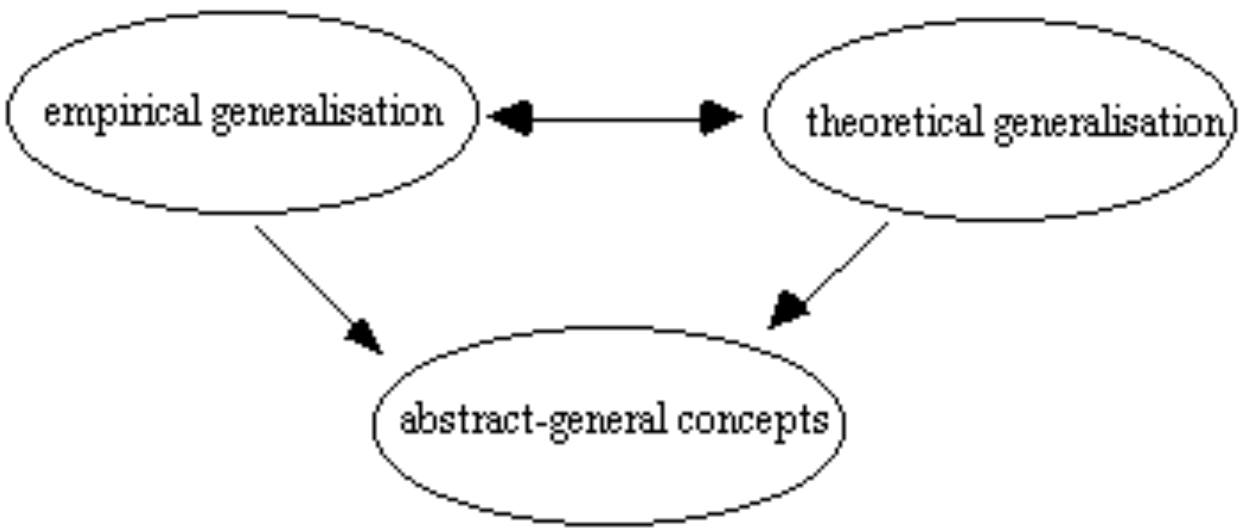

Figure 3: Outline of the theoretical framework

DISCUSSION

The opening paragraph of this paper identified definitions, procedures, theorems and proofs as the fundamental building blocks of mathematics. Research in light of the theoretical framework is now discussed for each of these three, as well as problem solving and a reflection on history.

Definitions

The way in which young children develop the concept of an angle (Mitchelmore & White, 1998) has recently been investigated. This research has brought into question the popularly used definition of an angle as an amount of turning. As mentioned earlier, the research showed that children do not readily interpret turns in terms of the standard angle definition. The basis for the turns context is the turning movement-intuitively well-known to young children and, even though it leaves no visible trace, adequately measured without the need to construct any lines. To link turning to the standard angle, it must first be re conceptualised as a rotating radius. Since this radius cannot simultaneously be in both the initial and final position, a second radius must then be constructed as a reference line. This is clearly a difficult task for most children. The suggested instructional sequence for teaching angle would:

• *start with examples*, by looking for similarities between those physical angle contexts which most clearly involve two lines (eg: corners and bent objects);

• *involve empirical generalisation*, by identifying angles in the environment and comparing angle sizes;

• *integrate theoretical generalisation*, with children would being encouraged to verbalise the common elements and to construct abstract angle models which embody the commonalities, but with no formal definition of angle being imposed;

• *teach for abstraction*, by investigating contexts in terms of the definition which children do not naturally interpret using the standard angle model, either because the lines are not clear (eg: rebounds) or because one line or both lines must be constructed (eg: slopes, turns).

Procedures

Our research based on this framework (what we call teaching for abstraction) has shown how crippling abstract-apart concepts learnt by the ABC method can be for students attempting to study algebra and calculus (White & Mitchelmore, 1996). In particular, because learnt procedures are based on the symbols used when learning them (usually x and y) and not on the process, students struggle when confronted with situations when the usual symbols are not visible or apply procedures to inappropriate visible symbols because those symbols were the ones they used to learn the procedure. The division of indices "*am* rule" shown earlier is an example of such abstract-apart, symbol based learning.

Results like "area of a rectangle = length x width" can in a sense be considered as theorems which have no formal proof. In school mathematics, they are a combination of concept and procedure. An elementary concept of area is obtained by counting the number of squares inside a figure. Then, by empirical generalisation, students can see that in the case of a rectangle the multiplication of length with width gives the correct result. Finally, using theoretical generalisation, the area inside a rectangle can be seen as rows of squares. The number of squares in one row is the same as the length and the number of rows equal the width. Hence the total number of squares is found by multiplying length by width.

Theorems and proofs

Theorems with proof usually first occur in the study of geometry. How writing proofs fits with abstraction and generalisation has not been well researched. However, the advent of dynamic geometry software now provides a tool allowing deeper analysis of the role of generalisation. Such software allows theorems to be constructed on the screen and then key features moved to show that the theorem holds in all types on configurations. For example, Figure 4 shows some 'stills' for the theorem that the angle bisectors of a triangle are concurrent at the centre of the inscribed circle.

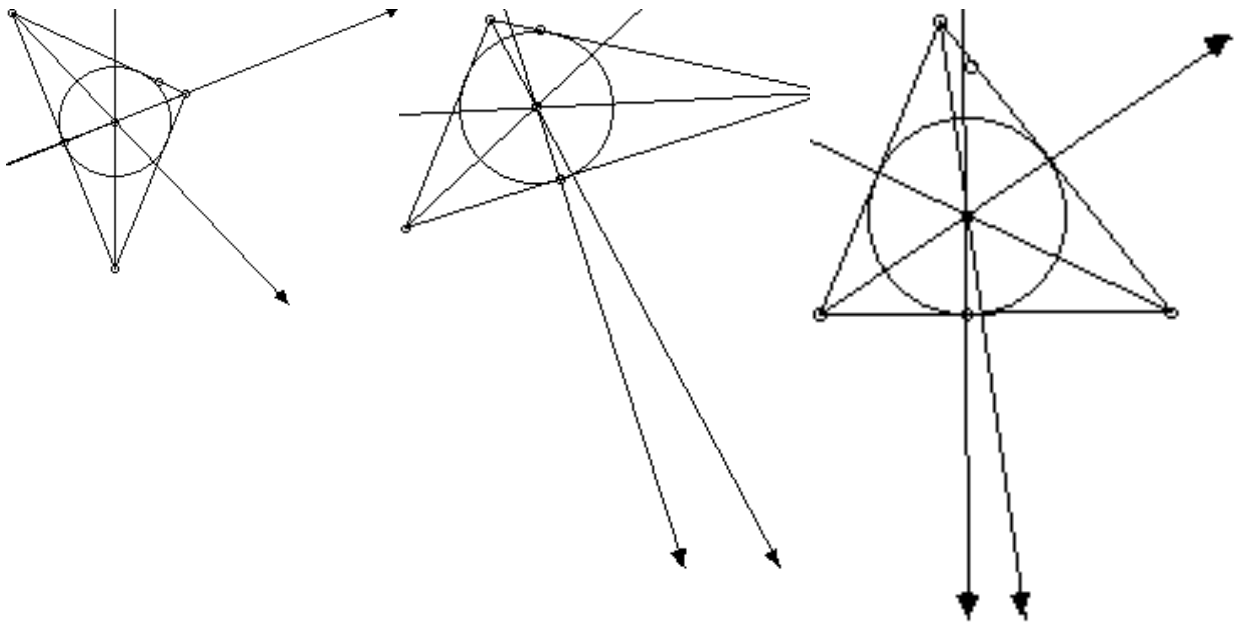

Figure 4: Dynamic geometry "stills"

Sharp and White (1997) showed the usefulness of such software for teaching geometry at a university level. This work has resulted in a current investigation into how such software might assist students in writing deductive proofs. Analysis suggests that constructing a diagrammatic version of a theorem on the screen and watching what happens when the configuration is moved around helps students visualise the theorem and understand the exact nature of what the theorem is saying. The standard student comment is "you have to think about the geometrical properties of the figure to construct them". In this sense, the software promotes theoretical generalisation by forcing students to think about the relationships between properties. The computer does not actually help with the construction of the formal proof, but the integration of the computer work with discussion in tutorials is of great assistance in proof writing. The visual display only involves empirical generalisation because the demonstration shows that the theorem works without indicating why. However, discussion provides the theoretical generalisation dimension and the two combined appear to provide a powerful teaching tool.

Problem solving

Another approach to learning advocated by many educators (and syllabus documents) is through problem solving. The tactic is to pose a problem, and then to leave students to solve it and convince others the solution is correct. Mitchelmore (in press) cites the example "*Can you tessellate the plane using a scalene triangle?*" In trying to solve this problem, all sorts of geometry concerning congruence, angles and parallels arise and become connected-in particular: angles on a straight line; the angle sum of a triangle; and corresponding, alternate, and co-interior angles formed by a line intersecting several parallel lines. Wrestling with this single problem produces a far deeper understanding of these concepts and results than teaching each one separately. We claim our framework supports the "problem-solving method" because problem solving clearly involves theoretical generalisation. Situations such as tessellations are analysed to identify the *structure* of the problem, the *basic*elements, and the *essential connections* between them. The concepts which are formed in this way are not isolated, but take their meaning from their relations to one another.

One way to promote problem solving is to use open-ended questions which require more than one answer. (Sullivan, Warren & White, 1998). For example, a standard closed question is: "A rectangle is 8m long and 5m wide. What are the perimeter and area of the rectangle?" The accompanying open-ended question is: "A rectangle has a perimeter of 30m. What might be the area of the rectangle? (Give at least 3 answers)". Research has shown that the latter is cognitively more demanding. On analysis, the closed question can be correctly attempted with an abstract-apart knowledge of two rules. However, the open-ended version requires perimeter and area to be linked by using the perimeter to find possible side lengths and then using these lengths to find the area. This process involves the recognition of alternative possibilities. Hence, it is more cognitively demanding because it requires analysis of the *structure* of the problem, the *basic* elements and the *essential connections* between them- a mix of theoretical generalisation and abstract-general concepts.

An historical failure

The framework also helps explain some of our past educational failures. For example, the "new math" was an abstract to concrete approach. The belief was that studying an abstract structure (sets) would result in identifying the isomorphism between set theory and the structure of numbers and thus develop number concepts. The result was that students set-theoretic knowledge was comprised largely of abstract-apart concepts and they did not make the desired transfer. We now know that it is imperative to begin number learning with concrete experience and to promote theoretical generalisation about the relationships between numbers and real life. Experience does not necessarily promote theoretical generalisation, which means the teacher is a key facilitator in the learning of abstract concepts.

CONCLUSION

We argue that the cognitive framework which integrates abstract-general concepts, empirical generalisation and theoretical generalisation for learning is a useful tool offering clear, practical guidelines which can be adapted to varying contexts. The cognitive focus of our arguments does not mean that we are down-playing the role of cultural and language factors. We believe that individual educators need to assess such factors in their local setting. Also, it is true that, in Australian and New Zealand schools, most curricula are structured to promote abstract thought and that students are likely to be far happier when provided some opportunity to succeed in their local setting.

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