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TRADITIONAL PEN-AND-PAPER VS MENTAL APPROACHES TO COMPUTATION: THE LESSON OF ADRIEN ®

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New mathematics syllabi are facing the issue of whether to discontinue the emphasis on traditional pen-and-paper algorithms and replace it with a focus on self initiated written algorithms, mental computation and number sense. Efficient and effective strategies for mental computation differ markedly from those that underlie traditional algorithms. They tend to be more wholistic and less reliant on separation into place values. These strategies often reflect the strategies required for estimation, and are more closely related to the spontaneous computational activity of children. This paper discusses traditional and mental approaches to computation in relation to the mental strategies for multiplication and division word problems employed by a child, Adrien, over a three-year period from 1993 to 1995 (Years 4 to 6). Although he was considered to be a higher ability student, Adrien was not a "lightning calculator", nor was he capable of such calculative feats as products of two eight-digit numbers. However, he was successful at multiplying and dividing two and three-digit numbers before such calculations were taught because he employed his own efficient and (it could be argued) advanced strategies that exhibited more number sense than the classroom taught traditional algorithms. His strategies exhibited both change and consistency and showed associated understandings. His performance highlighted the possibilities for computation syllabi where children are allowed to develop their own spontaneous strategies and indicated the disadvantages for syllabi, such as that still existing in Queensland, where traditional algorithms are still a major component.

In this paper mental computation is defined as arithmetic calculation without the aid of external devices (eg, pen and paper, calculator), with numbers greater than 10, and excluding number fact recall. Interest in mental computation as an important computational method is not new. However, its significance is now seen in terms of its contribution to number sense as a whole. To achieve this end, many researchers have argued that children need to develop proficiency in mental computation through developing their own self developed or spontaneous strategies rather than through memorisation of procedures (Kamii, Lewis, & Livingston, 1993; Reys & Barger, 1994). Further, the importance of mental computation as a personal construction was recognised in the National Statement (Curriculum Council & Australian Education Council, 1991) where it was stated

> *...students should be encouraged to develop personal mental computation strategies, to experiment with and compare strategies used by others, and to*

> *choose from amongst their available strategies to suit their own strengths and the particular context. (p. 109)*

In the present Queensland Mathematics Syllabus (Department of Education Queensland, 1987a), mental computation is only mentioned in the context of the mathematical process calculating:

> *In developing the ability to calculate, students will use mental algorithms, written algorithms, calculator algorithms... (p. 45)*

However, "mental algorithms" is not explained. In the *Curriculum Guidelines* section of the same document, it appears that mental strategies are to be used merely to "extend basic facts" (Department of Education Queensland, 1987a, pp. 16, 17). Moreover, mental computation is treated inconsistently throughout the Sourcebooks (these are support documents for the syllabus). In the Year 4 Mathematics Sourcebook (Department of Education Queensland, 1987b), specific mental strategies are mentioned for addition (eg, 52+29=52+30-1) and subtraction (eg, 82-29=82-30+1). Further, it is recognised that "different strategies can be used for one calculation" and "procedures for mental calculation do not necessarily follow the written algorithm" (p. 79). Unfortunately, the writers of the other Sourcebooks did not continue the groundwork established in the Year 4 Sourcebook. Mental computation is not mentioned again until the Year 7 Mathematics Sourcebook (Department of Education Queensland, 1987c), and only addition mental algorithms are explained. Incidentally, these same strategies have been reported as being employed spontaneously by children as young as 8 and 9 years old (Cooper, Heirdsfield, & Irons, 1996). Further, Dutch mathematics programs emphasise mental addition and subtraction in the lower grades (Beishuizen, 1993). The mental strategies for two-digit addition mentioned in the Year 7 sourcebook are taught to Dutch children in second grade.

As a consequence of an old and outdated syllabus, traditional pen-and-paper algorithms are still taught out of context in Queensland schools. As Cooper, Heirdsfield, and Irons (1996) reported, this has resulted in a tendency in Queensland children to use strategies for mental computation that reflect the procedures underlying the pen-and-paper algorithms regardless of their knowledge and ability to use more efficient strategies.

There is research evidence from reform classrooms that active involvement in mathematics learning enables children to understand computation, particularly where formal mathematical knowledge is built on informal knowledge (Carraher, Carraher, & Schliemann, 1987; Carroll, 1997; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, & Fennema, 1997; Kamii, Lewis, & Livingston, 1993; Thompson, 1994). Carroll (1997) and Kamii, Lewis, and Livingston (1993) documented the mental and written computational procedures invented by children who are active in their learning. They reported that children could produce a wide variety of efficient strategies that exhibit sound number understanding even though there was little direct teaching of algorithms. The children were encouraged to actively construct their own strategies by creating problems and explaining solutions.

There is research evidence that children can use self-developed strategies to efficiently and effectively solve mental multiplication and division problems of two or more digits, even before instruction (e.g., Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Kouba, 1989; Mulligan & Mitchelmore, 1997). Even in studies of children's solution strategies for more difficult multiplication and division word problems (Murray, Olivier, & Human, 1994), some self-developed strategies have been used (eg, repeated addition, decomposition and compensation for multiplication; and repeated subtraction, use of multiplication and partitioning for division). There is also evidence for the negative effect of traditional algorithm instruction on efficient mental strategies for multiplication and division examples. For example, Kamii et al. (1993) reported that 60% of third graders who had not been taught the traditional multiplication algorithm were able to mentally solve 13 x 11 (by thinking 13x10=130, 130+13=143); a problem which, in contrast, was only successfully mentally solved by 15% of fourth graders who had been taught, and therefore used, the traditional algorithm.

After more than 10 years, the Queensland syllabus is about to be rewritten. If the new syllabus is to reflect the worldwide reform of mathematics education, emphasis on traditional pen-and-paper algorithms should be replaced by a focus on self initiated written algorithms, mental computation and number sense. Therefore, it is essential to study how children think and how they develop mathematical understanding. This paper reports on one child's progress in using mental strategies for multiplication and division word problems through Years 4 to 6.

THE STUDY

Subject

Adrien was one of approximately 100 children chosen in Year 2 to participate in a large five year longitudinal Australian Research Council funded study into children's spontaneous mental strategies. The results for addition and subtraction in Year 2 to 4 are reported elsewhere (eg, Cooper, Heirdsfield, & Irons, 1996). Furthermore, Adrien's addition and subtraction strategies are discussed in Heirdsfield (1999). Multiplication and division results for the children in Years 4 to 6 are reported in Heirdsfield, Cooper, Mulligan, and Irons (1999).

Children were originally chosen by the teachers to represent one third above average, one third average, and one third below average. The children's ability levels were not revealed to the interviewer; however, Adrien's performance indicated that he was likely to belong to the "above average" group.

The Queensland mathematics syllabus advocates that children be introduced to the concepts of multiplication and division in Year 2, the multiplication symbol (up to 9x9=81) in Year 3. In Year 4 the standard written multiplication algorithm (2 by 1 digit) and the division symbol (up to 81¸ 9=9) are taught. The standard written multiplication written algorithm (2 by 2 digits) and the standard partition written division algorithm (2 by 1 digit) are presented in Year 5.

Interview procedures

The children were interviewed in the second and fourth terms of Years 4, 5, and 6. They were withdrawn from the classroom; videotaped interviews were conducted in a separate room. The instrument was Piaget's clinical interview technique. The children were presented with tasks, asked to solve them mentally, and directed to explain their solution strategies. The problems were presented in the form of pictures, and orally as the interviewer verbalised the task (eg, a picture of a calculator marked with $19 was shown to the child as the interviewer asked, "What is the total cost of 5 calculators?"). Adrien was presented with all tasks at all interviews.

Tasks

The tasks consisted of one, two and three digit multiplication and division word problems. The multiplication problems comprised equal grouping examples. The division tasks were represented by both partition and quotition word problems. The numbers were chosen in the hope of encouraging the use of spontaneous mental strategies (Table 1).

Table 1.

Number combinations for multiplication and division word problems

Multiplication	Division
5 x 8	24 ¸ 4 (partition)
3 x 9	60 ¸ 5 (partition)
7 x 50	100 ¸ 5 (partition)
5 x 19	200 ¸ 5 (partition)
10 x 40	450 ¸ 50 (quotition)
3 x 99	248 ¸ 8 (partition)
5 x 25	225 ¸ 25 (quotition)
4 x 26	198 ¸ 6 (partition)
19 x 25	168 ¸ 21 (quotition)
3 x 195	
7 x 45	

ADRIEN'S RESULTS

Children's self developed strategies for multiplication and division have been described elsewhere (eg, Harel & Behr, 1991; Kamii et al., 1993; Murray, Olivier, & Human, 1991, 1992, 1994). Although traditional pen-and-paper algorithms are based on sound reasoning and complex mathematical concepts, children (and even adults) generally follow a procedure as a set of rules (Kamii et al., 1993). In contrast, the self developed strategies reflect conceptual understanding of number and operation. Adrien's strategies (Tables 2 and 3) are discussed in relation to some of these associated understandings.

Overview

It was evident that Adrien had a good grasp of number facts, although it is interesting to note that he did not consistently recall facts as time progressed. As an example, 3x9 was solved using recall in the first, third and last interviews, and solved using a derived facts strategy in the other interviews. Further, 5x8 was not consistently solved using fact recall. For the division example 60¸ 5, a derived fact strategy was used in the last three interviews, yet fact recall was used in the first three.

More complex examples were also solved in a variety of ways, for example, 225¸ 25 (Table 3). To solve this example, Adrien used his knowledge of multiples of 25, but there were at least three different variations to the strategy. Further, strategy choice appeared to depend on the number combinations; for instance, the strategy Adrien employed for 3x99 was different from that used for 10x40. In contrast, a single strategy was at times used across examples; for instance, the same strategy was used to solve 3x9, 3x99, 19x25 and 3x195. As numbers close to multiples of 10 or 100 were involved in these calculations, it would make sense to use a similar strategy. Although Adrien may not have been able to verbalise the distributive law, it was this law of operations which he used intuitively to solve these examples and more, for instance, 4x26 (4x24 + 4). However, it is interesting (or disappointing) to note that Adrien resorted to a right to left separated strategy similar to the taught pen-and-paper algorithm for solving 3x195 and 5x19 in Year 6. In doing this, Adrien no longer verbalised the numbers in terms of place values; he treated the numbers as single digits in columns. He also adopted a strategy that exhibited less efficiency than the very powerful strategy that Adrien had used in earlier years.

Multiplication

Adrien's mental strategies for the multiplication tasks are described in Table 2.

Table 2.

Adrien's strategies for multiplication problems over 6 interviews

Question	1. Year 4 Term 2	2. Year 4 Term 4	3. Year 5 Term 2	4. Year 5 Term 4	5. Year 6 Term 2	6. Year 6 Term 4
5 x 8	number fact	1/2 of 10x8	number fact	5x10 - 5x2	1/2 of 10x8	1/2 of 10x8
3 x 9	number fact	3x10 - 3	number fact	3x10 - 3	3x10 - 3	number fact
7 x 50	7x5, add 0	8x50 - 50	7x5, add 0	2 lots of 50 make 100, 4 to 200, 6 to 300, 7 to 350	7x5, add 0	2 lots of 50 make 100, 4 to 200, 6 to 300, 7 to 350
5 x 19	5x20 - 5	1/2 of 10x19	5x20 - 5	5x20 - 5	5x9, 5x1, add 4.	5x9, 5x1, add 4.
10 x 40	Add 0	Add 0	Add 0	Add 0	Add 0	Add 0
3 x 99	3x100 - 3	3x100 - 3	3x100 - 3	3x100 - 3	3x100 - 3	3x100 - 3
5 x 25	4x25 + 25	4x25 + 25	4x25 + 25	4x25 + 25	4x25 + 25	4x25 + 25
4 x 26	4x25 + 4	4x25 + 4	4x25 + 4	4x25 + 4	4x25 + 4	4x25 + 4

19 x 25	20x25 -25	20x25 -25	20x25 -25	10 lots of 25 make 250, 20 to 500, take 25 ® 475	20x25 -25	20x25 -25
3 x 195	3x200 -15	3x200 -15	3x200 -15	3x200 -15	3x5, 3x9 add 1, 3x1, add 2	3x5, 3x9 add 1, 3x1, add 2
7 x 45	7x40, 7x5, add 280 & 35	7x40, 7x5, add 280 & 35	7x5, 7x4, add 3	7x5, 7x4, add 3	7x5, 7x4, add 3	7x5, 7x4, add 3

The one digit examples were solved by the use of number facts (either from memory or derived from another fact) and Adrien showed an ability to relate facts to the number 10 (eg, 5x8= 1/2 of 10x8; 3x9=3x10-3). Multiples of ten tasks (eg, 7x50, 10x40) were solved by using the multiples of ten principle (eg, 3x4=12 means 3x40=120). Tasks with numbers near easily used numbers were solved wholistically by relating the task to a similar one using the easier number (eg, 5x19=5x20-5, 3x99=3x100-3, 3x195=3x200-15). Where possible, the number 25 and the relation 4x25=100 were used (eg, 5x25, 4x26). In doing this, Adrien showed he was able to use the distributive law on both the first and second number. The only task that was not easily translated to an easy number or a fact, was solved by separating the larger number into place values.

There were interesting trends in Adrien's use of multiplication strategies as he progressed from Year 4 to Year 6. Strangely, Adrien's mental strategies were mostly as efficient and powerful at the start of Year 4 as they were at the end of Year 6, and in some cases the efficiency and power of his strategies declined. Interviews 4, 5 and 6 showed more construction of solutions from first principles. In three of the tasks (5x19, 3x195 and 7x45), Adrien's responses showed evidence that the teaching of pen-and-paper algorithms was having an effect on his choice of multiplication mental strategies. He gave up powerful and efficient wholistic strategies for a strategy that mirrored the traditional pen-and-paper algorithm procedure. He also seemed to lose meaning in terms of numbers. When he explained his strategy solution in the Year 4 interviews, it was evident he thought of the numbers in relation to place value. However, in later interviews the numbers became mere digits separated into columns; for instance "7x40" became "7x4", and 3 was added, rather than 30. Further, instead of calculating from left to right (as in earlier interviews), the procedure was completed right to left (as is the taught algorithm).

Division

Adrien's mental strategies for the division tasks are described in Table 3. Again, the simpler tasks were solved by the use of basic facts (memorised or derived) or by multiples of ten principles (eg, 12¸3=4 means that 120¸ 3=40 and 120¸ 30=4). More complex examples are solved by separating the larger numbers into place values (eg, 248¸ 8 is 240¸ 8 and 8¸ 8) or by using the inverse principle (168¸ 21 is ?x21=168). He also used a trial and error strategy in difficult cases.

The example, 168¸ 21 was particularly difficult for most children, although some children solved this counting in 21s or doubling, and also guessing and checking. Adrien used a guess and check strategy in the final interviews. This would appear to be an efficient strategy for such an example. Other strategies involved partitioning the dividend into known multiples of the divisor. A similar strategy was reported by Murray, Olivier, and Human (1994) in their study of fifth graders' self initiated strategies.

Table 3.

Adrien's strategies for division problems over 6 interviews

Question	1. Year 4 Term 2	2. Year 4 Term 4	3. Year 5 Term 2	4. Year 5 Term 4	5. Year 6 Term 2	6. Year 6 Term 4
24 ¸ 4	number fact	number fact	number fact	For 20 it's 5, for 24 it's 6	For 20 it's 5, for 24 it's 6	number fact
60 ¸ 5	number fact	number fact	number fact	For 50 it's 10, for 10 it's 2, 10+2=12	For 50 it's 10, for 10 it's 2, 10+2=12	For 50 it's 10, for 10 it's 2, 10+2=12
100 ¸ 5	10¸ 5, add 0	10¸ 5, add 0	number fact	number fact	number fact	5x20=100
200 ¸ 5	10 rooms hold 20, so 5 rooms hold 40.	10 rooms hold 20, so 5 rooms hold 40.	number fact	100¸5=20, so 200¸ 5=40	100¸ 5=20, so 200¸5=40	5x20=100, so 5x40=200
450 ¸ 50	45¸ 5	45¸ 5	45¸ 5	500¸50=10, -1® 9	500¸50=10, -1®9	45¸ 5
248 ¸ 8	8¸ 8=1, 240¸ 8=30, 30+1=31	240¸ 8=30, 8¸ 8=1, 30+1=31	240¸8=30, 8x1=8, 30+1=31	240¸8=30, 8¸8=1, 30+1=31	Just knew, couldn't explain	240¸ 8=30, 8¸ 8=1, 30+1=31
225 ¸ 25	10x25=250, 250-25=225, so 9.	10x25=250, 250-25=225, so 9.	200¸25=8, add 25 to make 225, so 9	8x25=200, +25=225, so 9	4x25=100, 8x25=200, +25, so 9	200¸ 25=8, add 25 to make 225, so 9
198 ¸ 6	180¸ 6 + 18¸ 6	180¸ 6 + 18¸ 6	180¸ 6 + 18¸ 6	180¸ 6 + 18¸ 6	180¸ 6 + 18¸ 6	180¸ 6 + 18¸ 6
168 ¸ 21	5x21=105, +21=126, +21, etc until reach 168.	5x21=105, 3x21=63, 5+3=8	160¸20=8, so try 8x21	160¸20=8, so 160¸8=20, and 8¸ 8=1	Guess & check by multiplying	Guess & check by multiplying

In terms of trends across the six interviews, changes were not as strong as for multiplication. However, it was still evident that the three years of growth and learning had not improved Adrien's strategy use markedly, and that there appeared to be slightly more derivation of facts in some tasks in later interviews. The evidence for a teaching effect was not as strong. There were some shifts of strategy towards the traditional pen-and-paper procedure but these were not clear-cut and did not have the same reduction in meaning in terms of numbers as for the multiplication tasks. Adrien tended to treat the numbers in division examples consistently as whole numbers. The task 248¸ 8 was originally solved by dividing the ones first (right to left), but Adrien progressed quickly into solving this problem by moving left to right which, it could be argued, is more efficient. It may also be argued that this strategy reflects the taught algorithm. However, at no stage did "240¸ 8" become "24¸ 8" as it does in verbalising the taught procedure. Similarly, for 198¸ 6, "180¸ 6" was never verbalised as "18¸ 6".

Summary

In summary, Adrien exhibited an intuitive understanding of place value, number facts, and number and operation. He understood multiplying by 10 (10x40), the division concept (450¸ 50), the distributive property (eg, 19x25, 3x195, 225¸ 25 in Interview 1), and inverse relationships (division examples were often solved by thinking multiplication). He tended to treat numbers as wholes, rather than breaking them up into single digits, and accessed a variety of strategies both during an interview and over the six interviews; that is, he exhibited flexibility. Further, different number combinations elicited different strategies. All examples were solved successfully over the six interviews. Some examples were solved with the same strategy throughout. Others were solved with slightly different strategies over the interviews. One could ask what Adrien had learnt in the three years, as the only conceptual changes appeared to be a trend towards the taught pen-and-paper multiplication algorithm for some examples. He appeared to have good higher intellectual functioning as evidenced in his ability to determine ways in which to relate tasks to simpler ones and choose effective solution strategies.

The types of strategies Adrien used reflected those used by children reported in such studies as Murray, Olivier, and Human (1992, 1994). The difference though is that Adrien's classroom instruction followed the Queensland syllabus. He was not part of a teaching experiment focusing on children's developing conceptually sound strategies "in a socially-supportive learning environment" (Murray, Olivier, & Human, 1992).

DISCUSSION AND CONCLUSIONS

The obvious point to begin discussion is that Adrien was an efficient and effective mental computer using wholistic methods before instruction in algorithms. Thus, instruction in algorithms had little to offer him, particularly in terms of conceptual development. Second, the strategies he used were powerful and reflected wholistic approaches to computation and numbers. But, unfortunately, he seemed to move towards less powerful strategies, that were similar to traditional pen-and-paper procedures, particularly for some multiplication tasks.

Adrien used some complex strategies. He showed that he understood how to intuitively use distributive, inverse and other principles in wholistic ways. This supported the position of Murray, Olivier, and Human (1994) who argued that the distributive property was the most useful for solving multiplication and division problems and pervades most self initiated strategies children use. Adrien showed strong number sense. His strategies often depended on number combinations and were conceptually advanced. He knew his basic facts and multiples of ten facts.

The implications of Adrien's experience are that instruction on computation should reflect the needs of children, many of whom, like Adrien, could compute effectively before instruction. As Romberg and Carpenter (1986) argued, pupils' existing knowledge should form the basis for building new knowledge. Therefore, teachers need to be aware of a student's existing knowledge. Adrien's effective computation was based on good number sense and a variety of strategies. Thus, teaching of computation should focus on the support and maintenance (and development, if necessary) of informal and self-developed strategies. Teaching should also focus more on number understanding, principles of numbers and operations, and effects of operations on numbers; teaching should encourage more variety. When children fail in computation, they are commonly required to practise algorithmic procedure. Adrien showed that effective computation can be based on good knowledge of numbers and arithmetic principles. It is ineffective to attempt to fix procedure when the problem is numbers or principles.

It is important to give more attention to mental computation, as it has practical importance and because of its value in developing number sense and higher order thinking. Mental computation needs to be viewed as "inventing and applying strategies that are idiosyncratic but appropriate for a particular problem, based on one's understanding of the basic features of the number system and of the arithmetic operation" (Verschaffel & De Corte, 1996, p. 120). It is not suggested that children should be taught mental strategies. After all, at present formal written algorithms are taught, and yet, children are far from successful (McIntosh, 1991). Strategies based on the traditional written procedures are hardly efficient for examples of the types presented to Adrien over the three years. The case could be similar for any alternative mental strategy chosen to replace traditional procedures; a taught strategy will always not meet every task effectively nor meet the variety of approaches children are naturally drawn to. It would be, similar to the traditional procedures, an imposition on many children's naturally effective methods for computing.

Children should be encouraged to invent their own computational procedures, as they develop better understanding of the effects of operations on number, and place value. Further, children take responsibility for their own learning. Time should be spent on students' describing various solution strategies for problems, and these strategies should be valued. Children's discussions are useful for not only discovering their understandings, but also any misconceptions. Adrien showed that a syllabus encompassing mental computation would be a positive step forward if it also encompasses variety, individuality, number sense and arithmetic principles.

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