TEACHING 3x

Tom J. Cooper & Anne M Williams

Centre for Mathematics and Science Education, QUT, Kelvin Grove, 4059

Paper to be presented at the Annual Conference of the Australian Association for Research in Education, Researching Education in New Times, Brisbane Qld, 30 Nov - 4 Dec, 1997.

A common misunderstanding in early algebra is related to binary algebraic expressions such as 3x. This simple algebraic expression represents the generalisation that 'any number' has been multiplied by 3. However, it is syntactically similar to the arithmetic notation for two digit numerals (e.g., 32) and different from the arithmetic notation for a particular number multiplied by 3 (e.g., 3 x 2). Thus, many students believe that 3x represents a 3 beside a variable and write 32 when asked to substitute 2 for x.

This paper reports on the findings with respect to 3x from two teaching interventions to introduce algebra to grade 8 classes in a middle class state secondary school. The interventions related algebraic representations, concepts and principles to arithmetic representations, concepts and principles (see Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, & Mutch, 1997) via the development of informal generalisations. Establishing the meaning for 3x was a major component of both interventions. The expression was considered in five ways: (i) modelled by cups and counters; (ii) developed from patterns (e.g., 3, 6, 9, 12), transformations or function machines (e.g., 2 ---x3--> 6) and relationships (e.g., 2-->6, 8-->23, 5-->15); (iii) considered as repeated addition x + x + x (see Linchevski & Herscovics, 1996); (iv) used in more complex expressions (e.g., 3x+2 and 3(x+2); and (v) extended to relations such as 3 x 2x = 6x. The classes were videotaped, written materials collected, and selected students interviewed.

The paper provides background theory and models, outlines the teaching episodes relating to 3x, describes the students' reactions to this instruction, and discusses the students' understandings of 3x as an algebraic expression in relation to the five ways above. It explores the successful and unsuccessful teaching episodes and attempts to explain, for binary algebraic expressions of the type 3x, the relationship between instruction, prior knowledge and learning.

A major focus of recent research into the learning and teaching of algebra has been the transition from arithmetic to algebra. According to Booth (1988), Herscovics and Linchevski (1994) and Filloy and Rojano (1989), the difficulties and obstacles to developing algebraic concepts are caused by a cognitive gap between the knowledge required to solve arithmetical equations by inverting or undoing (backtracking), and the knowledge required to solve algebraic equations by operating on or with the unknown. In particular, algebra requires the application of properties and conventions of operations, not the operations themselves (Herscovics & Linchevski, 1994).

This research has lead to two outcomes. The first has been the proposal of a "pre-algebraic" level between arithmetic and algebra. The second has been some recategorisation of what is algebra. For example, the notion of variable as an unknown, or the solution of equations like ax+b=c by the usual method of backtracking or systematic approximation (Linchevski and Herscovics, 1996) are not considered to be algebra: They are considered to be part of the pre-algebra transition. The following section discusses these issues.

Background

Algebra is an abstract system in which members interact to reflect the structure of arithmetic. Hence, algebra can be considered as the generalisation of arithmetic, where its activity is based on a variable which represents a generalised number (any number).

Within school mathematics, algebra deals with expressions and equations, for example simplification of expressions, solutions of equations, substitution within expressions, and the relation of these to real-world situations. Expressions and equations are composed of numbers, operations, variables, and (for equations) equals. It would seem evident that these components have to be understood alone as well as together in expressions and equations.

Cooper, Boulton-Lewis, Atweh, Pillay, Wilss, and Mutch (1977) have summarised research findings with respect to these components. The operational laws that have significance for algebra are the commutative, associative and distributive laws, inverse, and order convention (e.g., Bell, 1995; Demana and Leitzel, 1988). Misconceptions with respect to these properties may lead to a "conceptual obstacle" in algebraic understanding (Bell, 1995; Herscovics & Linchevski, 1994). Variable has four meanings: generalisation, unknown, relationship, and member of abstract system (Usiskin, 1988). The first three meanings can be replaced by the fourth when expertise is gained, and the generalisation and relationship meanings are more advanced than the unknown meaning (e.g., Chalouh & Herscovics, 1988; Sfard & Linchevski, 1994). Collectively, these meanings reflect Kucheman's (1981) sequence of six levels for students' understanding of variable: as a number, without meaning, as an object, as a specific unknown, as a generalised number, and as an abstract variable. Equals means that both sides of an equation are equivalent and that information can be processed from either direction in a symmetrical fashion (e.g., Kieran, 1992; Linchevski, 1995). Research has identified two limited meanings that students have for equals: (a) syntactic indicator, a symbol indicating where the answer should be written; and (b) operator sign, as a stimulus for action or 'to do something' (e.g., Behr, Erlwanger & Nichols, 1980; Denmark, Barco & Voran, 1976; Filloy & Rojano, 1989). As well, equals can be viewed: (a) in relational terms as balance, for example, 2 and 3 balances 5; and (b) in transformation terms as two-way change, for example, 2 changes to 5 by adding 3 and 5 changes to 2 by subtracting 3 (Cooper & Baturo, 1992). Thus, viewing expressions and equations in terms of the above components is a complex undertaking.

Recent research shows that current instructional practices may not be as effective as previously thought. Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, and Mutch (1997) have summarised research findings with respect to two instructional approaches for teaching algebra (patterning and materials). Using patterns to construct generalisations appears to be an appropriate consequence of seeing algebra as the generalisation of arithmetic. However, research evidence is showing that patterns do not easily generate algebraic rules and do not lead to algebraic understanding (MacGregor & Stacey, 1995). Materials such as cups for variables, counters for numbers, and a balance beam for equals have been recommended by Quinlan, Low, Sawyer and White, (1993), Sowell (1989) and Thompson (1988). However, there have been criticisms of the effectiveness of materials: (a) concrete representations impose additional cognitive demands (Halford and Boulton-Lewis, 1992); (b) physical models contain intrinsic restrictions (Behr, Lesh, Post, & Silver, 1983); and (c) connections between materials and symbols are limited (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, & Mutch, in press; Hart, 1989).

As Boulton-Lewis et al. (in press) have reported, achievement rates in algebra have been poor (e.g., Brown, Carpenter, Kouba, Lindquist, Silver, & Swafford, 1988) and difficulties in learning algebra have long been documented (e,.g., Thorndike, Cobb, Orleans, Symonds, Wald, & Woodyard, 1923). Booth (1988) categorised algebra errors as including the non-numerical nature of algebra answers, and misconceptions concerning the meaning of letters and variables. Herscovics and Linchevski (1994) identified difficulties with equations and the equals sign. Further, instruction does not seem to be bridging the gap between arithmetic and algebra, particularly in developing meaning for variables and for the equals sign. In particular, instruction should prevent the common misconceptions: (a) substituting 2 for x in 3x results in the two-digit number 32; (b) not viewing equals as equivalence in simplifying the expression on one side of an equation; and (c) representing the situation in which there are 6 students for every professor by the equation 6s=p.

To overcome the above problems, Boulton-Lewis et al. (1997) proposed that the solution of a complex algebraic operation is the end product of a learning sequence of mathematical concepts that includes: binary arithmetic operations; complex arithmetic operations (a series of operations on numbers); and binary algebraic operations. They constructed the two path model in Figure 1 to represent the sequence of learning necessary for understanding complex algebra.

BINARY ARITHMETIC ------------------

| |

| V

| COMPLEX ARITHMETIC

V |

BINARY ALGEBRA |

| |

| V

--------------------> COMPLEX ALGEBRA

Figure 1. Model for learning complex algebra.

The model means that: (a) understanding binary operations such as 2x5 and 5+3 is a prerequisite of 2x and x+3 (binary algebra) which, in turn, are an important prerequisite to understanding 2x+3=11 (complex algebra); (b) understanding operational laws should also be applied to series of operations as well as individual operations; and (c) that learning complex algebra will be facilitated by understanding similar (isomorphic) structures in complex arithmetic. Hence, the understanding of arithmetical structure becomes an important component of learning algebra.

In addition to the implications of the model, Boulton Lewis et al. (1997) argued for: (a) accessing algebra through solving problems with one unknown (i.e., focusing on equals and unknowns to cross the cognitive gap); and (b) sequencing instruction through binary operations (one unknown), series of operations (numbers and one unknown), multiples of the unknown, acceptance of lack of closure and immediate solution, and relationships between two variables. This research therefore recognises a "pre-algebraic" level between arithmetic and algebra by reflection on what is algebra.

The Interventions

In an attempt to study the transition to algebra, a group of educators at QUT has undertaken an ARC funded longitudinal study of Years 7 to 9 students (Boulton-Lewis et al., in press, 1997; Cooper et al., 1997). As an extension of this study, interventions were undertaken with two Year 8 classes in March and May (10 lessons in each intervention).

The basis of the interventions was to see algebra as dealing with principles that are generalised from arithmetic, that is, as dealing with abstract schemas (Ohlsson, 1993). For example, the distributive principle holds for whole numbers, 2 x (3+4) = (2x3) + (2x4), decimal numbers, 4 x 4.7 = 4x4 + 4x0.7, mixed numbers, 5 x 3 2/11 = 5x3 + 5x2/11, and algebra 4(x+y) = 4x + 4y. Thus the distributive principle is an abstract schema because its meaning lies in terms of relationships not the particular content (whole numbers, fractions, etc.).

The interventions were informed by an analysis of understanding undertaken by Baturo and Cooper (1997) which concluded that mathematics teaching should involve: (a) modelling, the use of materials and language in situations that can structurally map to required mental representations or symbols (Hiebert & Carpenter, 1992); (b) reversing, the technique that ensures that relations and changes move in both directions, that is, going from multiplication (23x2) to answer (46) and answer to multiplication (what multiplied by 2 equals 84?), or going from algebra to real-world situations and real-world situations to algebra; and (c) variety, the use of a wide range of approaches, situations and materials (particularly to introduce variable). The interventions were also informed by the work of Kaplan, Yammamoto and Ginsburg (1989) who found that effective formal mathematical knowledge is always underpinned by informal mathematical notions.

Unit plans

The interventions were designed to be two ten lesson programs (called Unit 1 and Unit 2), the first in March and the second in May. The programs were given to two Year 8 classes. The intention was to cover the development of the notion of variable and the manipulation of variables in expressions and equations. To do this, the following teaching sequence was developed: (a) generalities of binary arithmetic; (b) arithmetical transformations, unknown-beginnings problems and backtracking; (c) generalities of complex arithmetic; (d) unknowns, patterns and relationships; (e) concept of variable and binary algebra; and (f) arithmetical structure and the beginnings of complex algebra. This sequence was designed to follow the two-path model (from binary arithmetic to complex arithmetic to binary algebra and, finally, to complex algebra), to emphasise meaning and generalisation, and to follow the proposals of Boulton -Lewis et al. (1997). Each element of this teaching sequence is discussed below.

Generalities of binary algebra. This component was designed to include operations and equals and to focus on identifying generalities in informal and meaning terms (i.e., concepts, processes, principles and heuristics). The intention was that it be based on pupils' reflections on their primary practice and take a static or relation approach to operations. It was to include generalities encompassing: (a) the actions of addition (e.g., 'adding like things') and multiplication (e.g., 'multiplying everything'); (b) the actions of their inverses, that is, subtraction (e.g., 'subtracting like things') and division (e.g., 'dividing into everything'); (c) the equivalence meanings of equals (e.g., both sides are the same') and balance (e.g., 'what you do to one side you have to do to the other'); and (d) the notions of expression (e.g., 3x4) and equation (e.g., 3x4=12).

Arithmetical transformations, unknown-beginnings problems and backtracking. This section was designed to introduce the dynamic or change/transformation view of operations and show how such a view can lead to activities in which pupils create and backtrack changes involving sequences of operations. The component was to include: (a) the transformation representation of operations, expressions and equations (e.g., '2+3=5 can be considered as 2 changes to 5 by +3'); (b) the two-way reversible mapping meaning of equals (e.g., 2 --> 5 by +3 and 5 -->2 by -3) and its relation to the balance view of equals; (c) creation of arithmetical paths that involve sequences of operations and the analysis of the effect on paths of various changes in operational sequence (e.g., "same end" and "fixed end" problems); (d) relation of arithmetical paths to real-world situations and solutions to unknown-beginnings transformation problems (e.g., 'I am a number, I have been multiplied by 3, 4 has been added, I am now at 19, what number was I?') using trial and error; and (e) reversing the direction of arithmetical paths, relating these to real-world situations and solutions to unknown-beginnings transformation problems using the backtracking strategy.

Generalities of complex arithmetic. This component was planned to include sequences of operations, arithmetic laws and operation conventions. These were considered from both a relation and transformation approach. Similar to (1) above, it was designed to focus on identifying generalities in informal and meaning terms. It was to include: (a) equating relation and transformation meanings and representations of arithmetic informally and formally; (b) expressions (e.g., 3x4) and equation (e.g., 3x4=12) involving sequences of operations; (c) generalities encompassing inverse of a sequence of operations (e.g., 'you need to reverse the order as well as reversing the operations'); (d) generalities encompassing the commutative (e.g., 'it does not matter which comes first'), associative (e.g., 'you can associate the numbers in any order') and distributive laws (e.g., 'multiplication and division works on all the numbers'); and (e) generalities encompassing operation order conventions (e.g., 'you need to do the brackets before multiplication and division before the rest').

Unknowns, patterns and relationships. This component was planned to focus on constructing and determining number patterns and relationships. It was designed to look at these from both relation and transformation perspectives. It was to include: (a) considerations of arithmetical paths and how unknown-beginning problems can be represented with the english language and with equations involving letters for the unknowns; (b) generalising patterns, and considering how generalisations of patterns can be represented with the english language and with expressions involving letters; and (c) looking at relationships, and considering how they can be represented with the english language and with expressions involving letters.

Concept of variable and binary algebra. This was designed to look at variable from a variety of perspectives developed in earlier components. It was to involve translating situations to representations involving letters and including relation and transformation perspectives. It was planned to continually relate these representations back to real world situations (for one operation alone). It was to cover: (a) variable as unknown (translating one-step unknown-beginnings transformation problems to variable form and considering this form as a representation); (b) variable as generalisation of pattern (translating patterns to variable form); (c) variable as relationship (translating relationships to variable form); (d) modelling binary algebra using materials (e.g., cups and counters, envelopes); and (e) translating binary algebra to real world situations and reverse.

Arithmetical structure and the beginnings of complex algebra. This was planned to focus on the extension of the informal generalisations of binary and complex arithmetic to more formal procedures for algebraic expressions and equations. It was to involve simple sequences of operations. It was designed to cover: (a) expressions and equations in terms of unknowns, patterns and relationships (extension of (5)* above); (b) modelling of complex algebra (cups and counters, ribbons); (c) translating complex algebra to real world situations and reverse (including focus on student-professor problem); and (d) manipulation of expressions (simplification and substitution) and equations (one variable solutions), and how these manipulations differ.

Teaching sequences

The two units were designed so that Unit 1 covered binary arithmetic, transformations and unknowns, complex arithmetic, and unknowns, patterns and relationships; whilst Unit 2 covered variable and binary algebra, and arithmetical structure and complex algebra. The idea was that Unit 1 would attempt to have the pupils reflect on their experience of arithmetic to draw out generalities which appear to hold across the various types of numbers. It was hoped that these generalities would include those that differentiate between addition/subtraction and multiplication/division and those that underlie the procedures used in simplifications and equation solving in algebra. As well, it was intended that Unit 1 would look at the transformational approach to operations and solutions to unknown-beginnings problems, plus patterns and relationships. The idea here was to lay the foundations for variable as unknown and variable as generalisation and relationship, and to lead the pupils to seeing that letters could represent unknown and generalisation.

In following Unit 1, the plan was for Unit 2 to take up unknown-beginnings problems, patterns and relationships again and use them to introduce the concept of variable. This concept was to be reinforced by the use of materials to model variable. Special attention was to be given to binary algebra, particularly the meaning of expressions such as 3x. The procedures involved in developing the notion of variable were to be reversed to introduce substitution. The plan was for Unit 2 to be completed by using the generalisations from Unit 1 to simplify expressions and solve equations.

To achieve this, the plan was for Unit 1 to begin with a classroom activity to obtain a snapshot of pupils' abilities with respect to complex arithmetic. Then it was planned to move from binary arithmetic to arithmetical paths to complex arithmetic and, finally, to unknowns, patterns and relationships. The teaching aid was to be calculators. Unit 1 was to emphasise informal flexible notions about arithmetic that hold across the number types and can be generalised to 'letters'. The plan was for the lessons to be divided so that there were: (a) 2 lessons for generalities of binary arithmetic; (b) 2 lessons for arithmetical transformations, unknown-beginnings problems and backtracking; (c) 3 lessons for generalities of complex arithmetic; and (d) 3 lessons for unknowns, patterns and relationships.

The plan for the interventions had Unit 2 reviewing Unit 1 preparatory to introducing the notion of variable. From here, the intention was to introduce binary algebra and then complex algebra. There was to be a real effort to enable pupils to understand algebra in relation to real world situations. The plan was for the lessons to be divided so that there were: (a) 4 lessons for the concept of variable; (b) 2 lessons for binary algebra; and (c) 4 lessons for arithmetical structure and the beginnings of complex algebra.

This paper

One of the crucial steps in the development of algebra knowledge, which was to be introduced at the end of Unit 1 and consolidated in the middle of Unit 2, was understanding binary algebraic expressions such as 3x. This simple algebraic expression represents the generalisation that 'any number' has been multiplied by 3. However, it is syntactically similar to the arithmetic notation for two digit numerals (e.g., 32) and different from the arithmetic notation for a particular number multiplied by 3 (e.g., 3 x 2). Thus, many students believe that 3x represents a 3 beside a variable and write 32 when asked to substitute 2 for x.

This paper looks at the effectiveness of teaching binary algebraic expressions such as 3x in the interventions.

Method

The methodology adopted for the interventions was the Vygotskian teaching experiment. Forty minute lessons were planned and then implemented one at a time. The lessons were observed and videotaped. After each lesson, the students' responses were reviewed and plans changed appropriately for later lessons. After each unit, a sample of students was interviewed.

Subjects

The subjects for the interventions were two year 8 classes (denoted 8C and 8Y) of a suburban co-educational state secondary school in a middle class suburb of Brisbane. Both classes were given interventions of 10 forty-minute sessions in March (Unit 1) and May (Unit 2).

The subjects for the interviews were 14 students, 7 from 8C and 7 from 8Y. The students were chosen as per Table 1. They were selected at the end of the first interventions based on their performance in the interventions. It had been intended that there only be 12 students in the interview sample (4 on ability, 4 on homework, and 4 who appeared to be interesting). However, selection was difficult and, in two cases, 5 students were included. An equal number of males and females was selected, as shown by M and F in Table 1.

Table 1

Selection criteria for interview

ClassAbilityHomework performanceInteresting

The algebra course taught in Year 8 at the school consisted of four separate units of approximately 10 lessons each. The first two units (in March and May), which were replaced by the interventions discussed in this paper, consisted of: (a) patterning activities that lead to generalisation in March; and (b) activities simplifying and substituting into expressions in May. There appeared to be no development of variable, equals, expressions and equations between the March and May units.

Instruments

The instruments for the interventions were: (a) an initial survey of student understanding; (b) observations and field notes, (c) adhoc and unplanned individual and group interviews and notes, and (d) artefacts in the form of students' completed worksheets and homework sheets.

The initial survey was a worksheet titled 'Writing about mathematics'. It contained items covering: (a) the meaning of =; (b) the meanings of the operations; (c) operational laws such as inverse operations, distributive and associative principles, order convention of operations, and the meanings of boxes and letters in equations.

The interviews administered after the interventions were semi-structured. Interview 1 (after Unit 1) was designed to: (a) provide background on the students' beliefs and values; and (b) give feedback on the intervention. To this end, the first part of Interview 1 focused on personal feelings about mathematics, general attitudes to mathematics evident at the school, beliefs about how mathematics should be learnt, and beliefs about what makes good teaching; whilst the second part focused on the quality of the teaching and worksheets in the intervention.

Interview 2 (after Unit 2) was designed to diagnose the students' understanding of variable. Questions focused on: (a) the meaning of expressions such as 3x, 3x+1, and 3(x+2); (b) simplification of expressions; and (c) identification of expressions and equations in terms of real world situations.

This paper looks at students' responses in the first part of Interview 2, as well as those parts of the lessons, the initial survey, other worksheets and Interview 1 that reflect on understanding of binary algebra expressions such as 3x.

Procedure

For each intervention, ten lessons were planned by the researchers and vetted by the teacher of 8C and 8Y. Then, after modifications, the lessons were taught in sequence by one of the researchers. The lessons were observed by the other researcher and the usual class teacher, and videotaped. At the end of each lesson, the children's responses were discussed by the two researchers and, where possible, the teacher; and the next lessons in the sequence modified to take account of the responses. Since 8C and 8Y followed a different timetable, it was possible to use students' responses in 8C to modify the same lesson to be given to 8Y and vice versa.

For most lessons, worksheets to support discussions and material activity were designed and given to the students for consolidation activities. Those not completed in class were set for homework. All worksheets were collected after being completed by the students. All homework sheets were collected at the next day's session. Completion of homework was not enforced (as this was normal practice at the school), so only those students with an interest completed worksheets outside of class time.

At the end of Unit 1, observations of the ten lessons were collated, along with the initial survey responses and the collected work of the students. The 14 students were selected for interview. For Interviews 1 and 2, the students were interviewed by one of the researchers before school or at lunch time in the same spare class room within two weeks of the end of the unit. These interviews were videotaped. Before beginning Unit 2, the interviews were analysed, and all results, lesson observations, and responses examined. As a consequence of this, the plan for Unit 2 was modified.

Across the two units, binary algebra expressions such as 3x (and equations containing such expressions) were considered in five ways: (a) modelled by cups and counters; (b) developed from patterns (e.g., 3, 6, 9, 12), transformations or function machines (e.g., 2 ---x3--> 6) and relationships (e.g., 2-->6, 8-->23, 5-->15); (c) considered as repeated addition x + x + x (see Linchevski & Herscovics, 1996); (d) used in more complex expressions (e.g., 3x+2 and 3(x+2)); and (e) extended to relations such as 3x+2x = 5x and 3 x 2x = 6x.

Results

The videotapes of the lessons were summarised and combined with the field notes of the observer-researcher and the notes of ad hoc interviews to provide a narrative description of the lessons. Overall class responsiveness was gained from reviewing the videos, from categorising and collating responses on the initial survey, and examining the collected worksheets. A profile of individual students was gained from transcribing and summarising interviews, combined with the survey and worksheet summaries. All responses were then related to the lesson descriptions. The focus of the analysis was the relation between teacher action and student response in terms of evidence of learning. The analysis was cumulative in that it occurred during and after each unit.

Unit 1 lessons

The beginning plan was for Unit 1 to cover 4 components: (a) binary arithmetic, (b) transformations and unknowns, (c) complex arithmetic, and (d) unknowns, patterns and relationships. In this plan, the first lesson was the initial survey. The fourth part of this survey focused on the meaning of boxes and variables. It particularly asked students the meaning of 3x. Of the 47 students that answered this part of the survey, 21 said that the x stood for a missing number, while 17 said that they did not know what it meant. One student responded that the x in 3x meant a negative number, another thought it meant an action ("do something"), and yet another student thought it meant multiplication ("times"). Five responded that 3x meant place value (means "thirty-something"). Only 1 student said that the x meant "any number".

The initial survey took one complete lesson for the students to complete. Therefore, a decision was made to reduce the time to be spent on the first component of Unit 1, binary arithmetic. As a consequence, although a lesson was spent on the meaning of equals, no time was spent on the meanings of the four operations in simple binary situations (e.g., 2+3, 5-2, 2x3, 12/3). This was because it was felt that Year 8 students should understand this and that it may jeopardise the teaching if such simple material began the units. Hence, no time was spent reinforcing the meaning of multiplication in one digit examples, for example, discussing 3x4 as "three lots of four" or 4+4+4. Instead time was spent on:

(1) differences between operations in more difficult binary arithmetic situations, for example, looking at the difference between 24+3, which is completed by "adding like things", and 24x3, which is completed by "multiplying everything", (and, in this, beginning to informally introduce the associative and distributive laws); and

(2) the difference between expressions (34+58) and equations (34+28=31x2) and how they can be changed, for example, looking at how expressions such as 34+28 remain the same by "doing and undoing" (i.e., 34+28 remains as 34+28 if multiplied by 10 and then divided by 5 and then 2) and equations such as 34+28=62 remain the same by "doing the same to both sides" (i.e., 34+28=62 remains a correct relationship when both sides are multiplied by 17, i.e., 34+28x17=62x17) .

The equals lessons appeared successful to the teacher-researcher and observer-researcher, but the expression and equation lessons were considered to be unsuccessful. Few students could successfully complete the worksheets that followed the expression and equation lessons, and discussions with groups of students during the worksheet activity indicated that there was little understanding of what was expected, nor of the differences between expressions and equations. To overcome this and to continue through the unit, the second component, arithmetic as a transformation, was introduced (with apparent success as observed by the researchers), and used to look at expressions and equations in terms of how they can be changed. The students appeared to like and understand the teaching of transformational arithmetic. As well, the second attempt at teaching the difference between expressions and equations appeared to be more successful than the first attempt. However, observations of worksheet activity continued to show a significant proportion of the classes exhibiting little understanding of the expression-equation difference. Therefore, a decision was made to relate expressions and equations to real world instances, as shown in Figure 2 below. This lesson appeared to be very successful. The students eagerly and correctly completed the worksheets.

Expression 5x4 There were 5 packets with 4 chocolates in each packet.

"Sum" 5x4= There were 5 packets with 4 chocolates in each packet - how many chocolates?

Equation 5x4=20 There were 5 packets with 4 chocolates in each packet - this made 20 chocolates.

Equation 5x4=18+2 There were 5 packets with 4 chocolates in each packet - this was the same number of chocolates as John who originally had 18 and was given 2 more.

Figure 2. Expressions and equations.

However, the time spent on lessons so far was greater than what had been planned, and so the next two components, (c) complex arithmetic, and (d) unknowns, patterns, and relationships, could not both be implemented within the school's timetabled two-week March program for algebra. At this point, the teacher intimated that she would like variable to be introduced in Unit 1 because this was the normal end point of this part of the school's curriculum. Thus, teaching moved to the fourth component (unknowns, patterns and relationships), and these three approaches to introducing variable were taught as in Figure 3.

Unknowns Patterns Relationships

3 ---- x3 ---> 9 Term - 1, 2, 3, 4, ..., n 8 ---> 24

11 ---- x3 ---> 33 Value - 3, 6, 9, 12, ..., 3n 13 ---> 39

? ---- x3 ---> 21 4 ---> 12

x ---- x3 ---> 3x=21 y ---> 3y

3 -- x3 -> 9 --+2-> 11 Term - 1, 2, 3, 4, ..., n 8 ---> 26

14 -- x3 -> 42 --+2-> 44 Value - 5, 8, 11, 14, ..., 3n+2 13 ---> 41

? -- x3 -> --+2-> 17 4 ---> 14

x -- x3 -> 3x --+2-> 3x+2=17 y ---> 3y+2

Figure 3. Three approaches for introducing variable.

Patterns and relationships were taught in the normal manner. For patterns, a sequence of numbers was generated and related to the term (directly, or by using a geometrical construction such as the number of matchsticks needed to make 5 squares in a row) and the question asked as to the nature of the nth term. For relationships, numbers were related to other numbers (either directly or through a "guess my rule" game) and generalisation sought for the relationship. However, unknowns were taught using transformations. Boulton-Lewis et al. (in press) and Herscovics and Linchevski (1994) have found that students solve binary algebraic equations such as 3x+4=19 by backtracking. This is best understood in transformation terms as Figure 4 shows. Thus, the unknowns approach was introduced using transformations.

4 ----- x3 ----> 12 ----- +2 ----> 14 transformation

? ----- x3 ----> ? ----- +2 ----> 20 unknown

6 <--- /3 ------ 18 <---- -2 ------ 20 backtracking

x ----- x3 ----> 3x ----- +2 ----> 3x+2 variable as unknown

Figure 4. Transformations and unknowns.

Of the three approaches, the transformations approach appeared to work best, but there were not enough lessons to really consolidate any approach. The students had great difficulty finding the generalisation in the patterning and relations activities (particularly when there were two operations). Teaching was modified to give hints as in Figure 5.

3 ------> 7

12 ------> 25 (Hint: Multiplication by 2,

7 ------> 15 adding 1)

m ------> ?

Figure 5. Example of relations activity with two operations.

Cups and counters were used with the three approaches to model 3x, x+3, 3x+2 and 3(x+2). Initially, this did not appear to be as successful as expected. The students tended to model 3x and x+3 with the same material (3 counters for the 3 and a cup for the x), and to see multiplying by 3 and adding 2 as the same as adding 2 and then multiplying by 3 (both 3 cups and 2 counters). The students showed that they did not have a facility with multiplication in terms of repeated addition. It had to be continually stressed that x+3 was an unknown with 3 added to it, whole 3x was three unknowns or unknown+unknown+unknown. Extra work was spent on this understanding and on modelling with cups and counters and the students seemed to show improvement, particularly in class 8C, but there was not the time for follow up.

Unit 2 lessons

As a result of what had been done in Unit 1, Unit 2 now had to cover complex arithmetic as well as binary and complex algebra. Originally, the plan for Units 1 and 2 had been to focus on binary and complex arithmetic in Unit 1 and then on binary and complex algebra in Unit 2. However, the major focus of the units was to translate general informal notions from arithmetic into algebra principles, and the problem with this original plan was remembering the arithmetic notions from Unit 1 to Unit 2 where the algebra was to be developed. Therefore, the decision in Unit 1 to continue unknowns, patterns and relationships on to variables and algebra was particularly useful, because it showed the importance of keeping similar arithmetic and algebra notions in the same lesson. Therefore, the plans for Unit 2 were modified to: (a) revise binary algebra; and (b) relate complex arithmetic and complex algebra notions. To maintain some consistency with what other Year 8 teachers were doing at the school, the second part of Unit 2 was restricted to simplification and substitution activities.

The first lessons in Unit 2 revised the Unit 1 introduction to variable using the unknowns, patterns and relationships approaches and cups and counters, and related both arithmetic and algebraic expressions and equations to real world situations. This was followed by another focus on the differences between expressions and equations in how they stay the same ("doing and undoing" for expressions and "doing the same to both sides" for equations).

After this introduction, extra work was spent on seeing multiplication as repeated addition and arrays (the area model) and translating the patterns in arithmetic into algebra. For example, 5x7 as 7+7+7+7+7, five tens as 10+10+10+10+10, and five unknowns as unknown + unknown + unknown + unknown + unknown, is translated to 5a as a+a+a+a+a; whilst the area model is illustrated as in Figure 6.

o o o o

3x4 = 12 as o o o o and 3cm by 4cm as 12 cm2 is translated to 3x x 4x = 12 x2.

o o o o

Figure 6. Area model translation arithmetic to algebra.

This work plus suitable examples from arithmetic was used to introduce simple simplification. For example, 2b + 3b = 5b and 2 x 3b = 6b follows on from 2 tens + 3 tens giving 5 tens and 2 lots of 3 tens giving 6 tens, whilst 2b x 3b = 6b follows on from 2 tens x 3 tens giving 6 hundreds. These simplifications were also modelled with cups and counters. In fact, the worksheets contained modelling activities (e.g., 2 unknowns + 3 unknowns = ? and U U + U U U = ?) before the algebra simplifications is requested. Time was spent on modelling and relating informal generalisations expressed in common language to algebra symbols (e.g., sum of 5 and number was 5+n, twice number subtract 7 was 2m-7). From observations, the impression was that the students were comprehending the meaning of variable and the use of the cups and counters much better than in Unit 1. One class (8C) appeared to grasp the differences between multiplying and then adding, for example, 3x+2, and adding and then multiplying, for example, 3(x+2).

Lessons then continued into two variable work. This was initially achieved vertically, related to arithmetic, and modelled by cups and envelopes (representing different unknowns - see Figure 7). Then, vertical and horizontal arrangements were related, with the horizontal setting out prevailing.

4 tens + 3 ones 4 weeks + 3days 4 + 3 sevenths 4x + 3y

+ 5 tens + 2 ones + 5 weeks + 2 days + 5 + 2 sevenths + 5x + 2y

9 tens + 5 ones 9 weeks + 5 days 9 + 5 sevenths 9x + 5y

Figure 7. Vertical development of two-variable simplification.

Observations indicated that the students received the early lessons here exceptionally well and appeared to understand the generalisations from arithmetic to algebra. However, as simplification of algebra contains subtraction and division as well as addition and multiplication, much of this could not be modelled. Instead, instruction relied on extending addition and multiplication work by appealing to the patterns that had already been set up, and on asking students to make common sense transfers (e.g., 3 x 4a = 12a, therefore, 12a / 3 = 4a). However, due to lack of time, some of the extensions were rushed and appeared not to be well grasped, particularly the two in Figure 8.

21 / 7 = 3 4x + 5y

21a / 3 = 7a + 3x - 6y

21mn / 7m = ? ? ? ?

Figure 8. Problem activities for the students.

More time needed to be spent on the two variable multiplication and division operations, as well as negatives in two variable simplification.

The simplification connections between algebra and arithmetic took much of the 10 lessons. Little time was left to spend on substitution, and two of these lessons had to be left to the teacher when changes in timetable prevented the researchers from attending two lesson times. However, for the little over one lesson undertaken by the researchers, substitution was modelled as the reverse or undoing of the methods used to develop the notion of variable, as in Figure 9.

2 ----> 12 2 ----> ??

12 ----> 42 rule: +2, x3 algebra: 4z-1 13 ----> ??

3 ----> 15 generalisation: 3(y+2) rule: x4, -1 4 ----> ??

7 ----> 27 28 ----> ??

Figure 9. Substitution as reverse of variable development.

The use of inverse in substitution appeared to work well but not enough time could be spent in verifying the first responses.

Interviews

Interview 1 (following Unit 1) supported the initial observations of lessons in revealing students lack of interest in the variables part of Unit 1. With regard to the researchers' teaching, it revealed that 9 of the 14 students thought that the cups and counters were a negative part of the teaching. With regard to the worksheets, the students said that their favourite worksheets were those concerned with equals, transformations and relating expressions and equations to real life situations. No more than 3 of the 14 students liked any of the worksheets concerned with introducing variables, with only 2 students liking more than one of the five worksheets being considered. This was equally the case for unknowns as well as patterns and relationships.

Interview 2 (following Unit 2) focused initially on students' understanding of 3x, 3x+1, and 3(x+2). The students responses were interesting. First, for 3x, 13 of the 14 students appeared to understand that this represented 13 multiplied by any number, 11 could transform the numbers 1, 6 and -3 by 3x (substitution), and 11 could represent 3x by cups and counters. However, only 6 could express 3x as repeated addition and 4 could represent a 3x pattern. Second, for 3x+1, 12 students appeared to understand that 3x+1 meant "3 times any number plus 1", 10 students could transform (substitute) and 12 could represent with cups and counters. Again, only 4 students could represent 3x+1 by x+x+x+1 and only 2 students could give a pattern for 3x+1. Third, for 3(x+2), 12 students appeared to read this with meaning. However, only 1 student could represent 3(x+2) as repeated addition (x+2) + (x+2) + (x+2), only 5 students could substitute, and only 4 students correctly used cups and counters.

Discussion and Conclusions

Initial analysis of this research has found a complex interaction between the understanding of arithmetic and the understanding of algebra. In particular, eight important findings emerged.

1. 3x as x+x+x: The findings supported Linchevski and Herscovics (1996) in stressing the importance of understanding 3x. The concept of 3x as repeated addition should have been sequenced in Unit 1. Although this omission was remedied in Unit 2, it was still a weakness of the students in examples which included more than one operation (e.g., interpreting 3(x+2) as (x+2) plus (x+2) plus (x+2)). It is particularly interesting to note that there appeared to be a relationship between understanding multiplication as repeated addition and correctly using cups and counters for multiplication situations such as 3(x+2).

2. Cups and counters: The use of materials (cups, counters) acted as a conduit between arithmetic notions. They appeared to work successfully in Unit 2 for binary algebra (e.g., 3x) and the simpler complex algebraic representations (e.g., 3x+1). Although it appeared useful for students to differentiate between multiplication then addition (e.g., 3x+1), and addition then multiplication (e.g., 3(x+2)) with cups and counters, understanding was not evident in Interview 2. This could reflect the criticisms of materials given by Halford and Boulton-Lewis (1992), Boulton-Lewis et al. (in press), and Hart (1989) that children see little connection between materials and symbols due to the additional cognitive demands that materials bring.

3. Usiskin's approaches: Usiskin's approaches (e.g., generalisation, unknowns, relationships, members of abstract system) seem important because of the way they reflect the different notions of generality. However, students appeared to have problems with patterns and relationships because they found it difficult to work out the patterns behind the general relationships when there was more than one operation. There was also difficulty with order of operations and the distributive law that required more time than could be given in these two teaching units. However, unknown as transformation was well received by the students and appeared in most cases to be understood. Therefore, as found by Herscovics and Linchevski (1994), variable as unknown and arithmetic transformation appear to offer good opportunities for the transition from arithmetic to algebra and seem to fit within the "prealgebraic" level between arithmetic and algebra proposed by Boulton-Lewis (1997).

4. Reversing and real world examples: The students really appeared to enjoy the work where real world situations were related to algebra. This seemed to provide an important grounding for the later work on simplification. Reversing the process for developing the notion of variable in relationships was also effective in teaching substitution, as was shown in the follow-up interviews. This seemed to be particularly useful in conveying the notion of equals (see Figure 2).

5. Informal notions (2 3/4 + 3 1/4 <--> 2X + 3Y + 3X +Y): This was a major part of both units. Teaching in arithmetic focused on developing informal notions such as "adding like things" and "multiplying everything". These notions were then translated into algebraic principles in simplification (see Figure 7). Lesson observations showed that students appeared to accept, enjoy and understand this transitional process from arithmetic to algebra.

6. Expressions and equations: One of the problems in upper secondary mathematics is students' use of incorrect procedures when making changes to expressions and equations in solution or simplification questions. For example, in equations, students will often divide only one side of the equation or subtract from only one side. In expressions, they will often perform an operation convinced that they have not changed anything. In Units 1 and 2, for both arithmetic and algebra, attempts were made to teach students that expressions should be changed by "doing and undoing" so that their value does not change. Similarly, equations should be changed by "doing the same to both sides". These attempts were not overly successful, perhaps because they were taught out of the intended sequence. Further work must be done in this area. It may be that more complex understanding is required to understand operations on expressions and equations.

7. Two path model: The two path model (Boulton-Lewis et al., 1997) appeared to provide a framework for effective teaching. However it should not be seen to consist of four separate steps to be performed across time. Rather, it represents a framework that should be followed for each separate notion and principle. For example, in simplification the following reflects the two path model: 3 ones + 2 ones, 3x + 2x, 3 tens + 4 ones + 5 tens + 2 ones, (3x + 4y)+(5x + 2y). Figure 7 shows part of this transition.

As a consequence, the teaching sequence should be changed to the following: (a) generalities of binary arithmetic; (b) concept of variable; (c) introduction of binary and complex algebra; (d) generalities of complex arithmetic; and (e) arithmetical structure and complex algebra.

8. Abstract Schemas: In reflecting on this research, it appears useful to consider algebraic principles as abstract schemas. Therefore the attempt to teach algebra should be seen as an attempt to teach abstract schemas. For instance, as Figure 7 shows, the process of simplification has to be extracted away from the particular instances in which they appear. For example, the tens and ones, weeks and days, wholes and parts have to be removed and replaced by arbitrary letters. This process is arduous for the learner, and the experience of the interventions showed that the process can be easily thwarted if it is complicated by a lack of understanding or lack of familiarity with components (e.g., multiplication as repeated addition, using cups and counters, order of operations in arithmetic).

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