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The architecture of mental addition and subtraction

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Research has shown that mental computation is a valid computational method which contributes to mathematical thinking as a whole (e.g., Sowder, 1990). It is also a process for which young children have exhibited a variety of proficient spontaneous strategies contrary to instruction (Cooper, Heirdsfield, & Irons, 1996a). This paper reports on a series of three studies on young children's understanding of mental addition and subtraction and describes the mental architecture of a proficient mental computer. Analysis of the first study showed that children's strategy use is idiosyncratic, but influenced by instructional emphases, experience and presentation forms; particularly in relation to the strategies underlying pen-and-paper algorithm procedures. Analysis of the second study identified a relationship between proficiency in mental computation, number fact knowledge and computational estimation. Initial analysis of the third study, which involves detailed construction of mental models, is indicating a more complex interaction.

The importance of developing number sense as an essential element of mathematics education has been recognised in the literature (e.g., Australian Education Council & Curriculum Corporation, 1991; Reys & Barger, 1994; Willis, 1992). Trafton (1992) suggested that people who possess number sense tend to solve computational problems by using knowledge about numbers, operations and their relationships. Mental computation may be viewed as a subset of number sense, as people who are good mental computers use "self developed strategies based on conceptual knowledge" (Reys, Reys, Nohda, & Emori, 1995, p. 324), and the ability to compute mentally is an indicator of the possession of number sense (McIntosh, 1995; Sowder, 1992).

In the past, the focus of primary mathematics computation has been the traditional pen and paper algorithms. However, more recent research has suggested that mental computation should play a more prominent role in number strands of mathematics curricula (e.g., Cobb & Merkel, 1989; McIntosh, 1996; Reys & Barger, 1991; Sowder, 1990; Willis, 1990). Reasons for its inclusion are: mental computation enables children to learn how numbers work, make decisions about procedures, and create strategies (e.g., Reys, 1985; Sowder, 1990); mental computation promotes greater understanding of the structure of number and its properties (Reys, 1984); and it has an utilitarian purpose (Clarke & Kelly, 1989; Maier, 1977). Further, Kamii, Lewis, and Jones (1991) recommended that children should be free to formulate their own mental strategies, as understanding of algorithms is improved if children construct strategies in line with their own natural ways of thinking. McIntosh (1996) also agreed that teaching mental strategies the same as formal pen and paper strategies have been taught in the past is not the solution to the present lack of attention given to mental computation.

It is within a framework of number sense (flexibility with and understanding of numbers and operations) that three studies of mental computation are analysed and reported in this paper. The first study charted Years 2 to 4 children's accuracy and strategy use for mental addition and subtraction (Cooper, Heirdsfield, & Irons, 1996a, 1996b; Heirdsfield & Cooper, 1996). The second study related Year 4 children's mental addition and subtraction proficiency with number fact knowledge and computational estimation proficiency (Heirdsfield, 1996). Finally, the third study aimed to generate a description of factors associated with Year 4 children's proficiency in mental addition and subtraction (Heirdsfield, in preparation).

Study 1

To explore children's mental addition and subtraction with respect to accuracy and strategy use, 104 children of varying mathematical ability (one third each of above average, average, and below average ability) were selected from 6 schools (representing a variety of social and cultural backgrounds). All their teachers followed the Queensland mathematics syllabus (Department of Education, 1991), where the focus of teaching number and operations is to develop basic number facts and the traditional pen and paper algorithms.

The children were interviewed three times in year 2, twice in year 3, and once at the beginning of year 4, using Piaget's revised clinical interview technique (Ginsburg, Kossan, Schwartz, & Swanson, 1983). The interview questions consisted of 2 and 3 digit addition and subtraction word problems, relating to money, and algorithmic exercises, all presented visually and read to the children. Further, there were three types of problems (based on Carpenter & Moser, 1984): join addition, separate subtraction, and missing addend subtraction. The children were withdrawn individually from the classroom and interviewed in a separate room, where the interviews lasted no more than 30 minutes and were videotaped. Paper and pencil and other calculating devices were not permitted. However, the children were allowed to count on their fingers.

Detailed results have been reported elsewhere (Cooper, Heirdsfield, & Irons, 1996a, 1996b; Heirdsfield & Cooper, 1996). In this paper, general results and trends will be discussed under three areas: general trends, word problems v. exercises, and additive and subtractive strategies for subtraction word problems and exercises.

Strategies

To analyse strategy use, a categorisation scheme (Cooper, Heirdsfield, & Irons, 1996a) based on Beishuizen (1993) was formulated. The resulting categories of counting, separation, aggregation and wholistic are described in Table 1.

Table 1

Mental strategies for addition and subtraction

Strategy Example

Counting 28+35: 28, 29, 30, .. (count on by 1)

52-24: 52, 51, 50, .. (count back by 1)

Separation right to left 28+35: 8+5=13, 20+30=50, 13+50=63

52-24: 12-4=8, 40-20=20, 28 (subtractive)

4+8=12, 20+20=40, 8+20=28 (additive)

left to right 28+35: 20+30=50, 8+5=13, 50+13=63

52-24:50-20=30, 2-4=2down, 30-2=28 (subtractive)

20+30=50, 4-2=2, 30-2=28 (additive)

Aggregation right to left 28+35: 28+5=33, 33+30=63

52-24: 52-4=48, 48-20=28 (subtractive)

24+8=32, 32+ 20=52, 28 (additive)

left to right 28+35: 28+30=58, 58+5=63

52-24: 52-20=32, 32-4=28 (subtractive)

24+20=44, 44+8=52, 28 (additive)

Wholistic Compensation 28+35: (28+2)+35=30+35=65, 65-2=63

52-24: 52-(24+6)=52-30=22, 22+6=28 (subtractive)

24+26=50, 50+2=52, 26+2=28 (additive)

levelling 28+35: 30+33=63

52-24: 58-30=28 (subtractive)

22+28=50, 28 (additive)

From the examples in Table 1, it is evident that the wholistic strategy

has less components and steps (less sub-problems) in its solution procedure than the aggregation strategy, which in turn has less components and steps than the separation strategy. The separation strategy requires each number to be separated into place-value components and which are separately added or subtracted and then recombined for the solution. As argued in Beishuizen (1993) and Wolters, Besishuizen, Broers, and Knoppert (1990), less components and steps means less cognitive load on memory during mental calculation. This should mean, in line with Sweller and Low (1992), that the aggregation and wholistic strategies should be more efficient and accurate particularly for more difficult examples. Similarly, left to right separation has less sub-problems than right to left separation and, therefore, should be more efficient than right to left separation in terms of cognitive load. Thus, the strategies right to left separation, left to right separation, aggregation and wholistic form a hierarchy of efficiency.

General trends

Over the 2 years, the percentage of children attempting the questions increased, as did the accuracy levels. This would be expected due to maturation. Further, addition was attempted with higher frequency and accuracy than subtraction. This finding reflects much of the literature which has reported the difficulties children have with subtraction (e.g., Fuson, 1984; Thornton, 1990).

Originally, counting was the dominant strategy; although the more difficult examples were successfully attempted with more advanced strategies. By year 3, left to right separation had become the dominant strategy. By the beginning of year 4, right to left separation was the dominant strategy (particularly for algorithmic exercises). As reported by Beishuizen (1993). separation strategies were more popular than aggregation strategies, although aggregation was more accurate. Overall, a variety of strategies was reported.

In Queensland schools, the traditional pen and paper algorithms for addition and subtraction are introduced in year 2 and further developed in year 3. The procedures for these algorithms are symbolic and follow a set pattern of activity: the numbers are written vertically with place values aligned, the place values are separated, the ones are operated on first, and ones and tens are regrouped as required, and computation proceeds right to left (from the ones to the tens to the hundreds). The right to left separation strategy may not involve carrying the ten in addition (e.g., 25+38: 5+8=13, 20+30=50, 50+13=63). However, the pen and paper procedure for addition and subtraction algorithms has common aspects with the right to left separation strategy for mental addition and subtraction (i.e., separate the ones and tens and work right to left by adding the ones and then adding the tens) and no other mental computation strategy has these commonalities. In fact the aggregation and wholistic strategies are in opposition to the pen and paper algorithm procedure in not separating all place values. Therefore, it is a reasonable assumption that the teaching of the pen and paper algorithms, with their attendant right to left procedure, will have an impact on children's choice of mental computation strategy. The results from the interviews show that, in year 3, there is strong increase in popularity for the right to left separation strategy (as Madell, 1985, also found). Children's responses to the word problems, and the relationship between these responses, is a strong indication that there is an instructional effect, that is, instruction in pen and paper procedures influences children to choose the right to left separation strategy for mental addition and subtraction.

Word problems versus algorithmic exercises

Word problems were attempted by more students and with greater accuracy than algorithmic exercises. This was interesting, as it does not reflect the sequence for teaching in Queensland schools. Generally, exercises are presented first; then, if the children are successful at this stage, they are introduced to word problems. The exception to this trend was in the last interview where the 3 digit exercises were attempted with greater accuracy and frequency than the equivalent word problems. Further, a greater variety of strategies was identified for word problems than for algorithmic exercises. At first, algorithmic exercises were attempted with a variety of strategies (though not as great a variety as for word problems), but this variety diminished in later interviews.

As argued above, in line with Sweller and Low (1992), the higher strategies (aggregation and wholistic) should be used more efficiently and accurately by children, particularly for more difficult examples, because they are the most cognitively efficient. The children's responses supported this in the early interviews, but not in the later interviews where the right to left separation strategy was the most highly used and most accurate. However, this result can be understood when familiarity and automaticity with procedures is included in the cognitive load equation. The instructional focus on the traditional pen and paper algorithms means that children become sufficiently familiarised with the right to left separation strategy procedures that it is cognitively efficient to use them in difficult examples. Hence, even in cognitive load terms, there appears to be an instructional effect due to the emphasis on traditional pen and paper algorithmic procedures in years 2 and 3.

Additive and subtractive strategies

Overall, the children exhibited both subtractive and additive strategies for separation and missing addend word problems. This was consistent with the findings of Carpenter and Moser (1984). Further, both strategies were employed for algorithmic exercises, also reported by Perry and Stacey (1994) for older students (years 8 to 12). Year 2 children predominantly used a subtractive strategy for separate problems and an additive strategy for missing addend problems, reflecting the semantic structure of the problem (similar to findings of Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993). However, by year 4, the strategy choice no longer closely reflected the semantic structure of the problems.

In contrast to findings of Carpenter and Moser (1984), Fuson (1986a, 1986b), Fuson and Willis (1988), and Secada (1982), there was not an emphasis on additive strategies. Once again, there appears to be an instructional effect as emphasis on the subtractive traditional pen and paper subtraction algorithm in Years 2 and 3 means that, by Year 4, most children had adopted this approach and discontinued additive procedures for subtraction problems.

Summary and implications

The major finding of study 1 is the instructional effects on mental computation of the teaching emphasis on traditional algorithms. The implication is that there should be less emphasis on teaching traditional pen and paper algorithms, more account of children's natural preferences and capabilities when designing curricula, and more emphasis on developing children's spontaneous strategies in problem solving environment.

However, questions still arise as to why some students are more accurate and flexible with strategies than others and how their expertise relates to their knowledge and performance in other mathematics topics. These questions were the motivation for study 2.

Study 2

During the early stages of study 1, it became apparent that, although a variety of strategies used by young children had been identified, some children by year 4 were employing one strategy consistently, while others employed multiple strategies. Research in the area of mental computation and number sense has identified computational estimation and number fact knowledge as contributing or associated factors (e.g., Hope & Sherrill, 1987; Resnick & Ford; 1981; Reys, 1984; Reys, Bestgen, Rybolt, & Wyatt, 1982; Sowder & Wheeler, 1989; Sowder, 1988). Thus, the focus of study 2 was to compare characteristics of multistrategy mental computers and unistrategy mental computers in relation to accuracy in mental computation, proficiency in computational estimation, and proficiency in number fact knowledge. Strategies for mental computation, computational estimation, and number facts were also identified.

The sample of 32 was drawn from the students involved in the longitudinal study, which was reported as study 1. They were chosen on the basis of accuracy and employment of a variety of strategies for mental computation. Each student then participated in two more interviews: computational estimation and number facts. Both mental computation and computational estimation tasks were presented in picture form, accompanied by printed numbers, and the problem verbalised by the interviewer. The number facts questions consisted of 8 addition and 8 subtraction facts to 20 (presented in written form).

The mental computation, computational estimation and number fact interviews were analysed with respect to strategy choice and accuracy. Number fact knowledge was also analysed with respect to speed.

Multistrategy versus unistrategy children

All students who employed multistrategies accurately were proficient at computational estimation and number facts. These students also employed wholistic strategies for both mental computation and computational estimation tasks, where possible. It was argued that the ability to use such strategies required a good understanding of number.

Thus number understanding was reflected in both mental computation and computational estimation. It has been argued elsewhere that the ability to compute mentally and estimate are related skills (Sowder & Wheeler, 1989). While not all computational tasks could have been calculated using wholistic strategies, the students who employed multistrategies accurately also used other efficient mental strategies (e.g., aggregation).

As well, number facts were recalled quickly and accurately. Where immediate fact recall was not employed in the number facts test, advanced derived facts strategies (DFS) were employed, for example, use doubles, use 10 (c.f., counting). All these students reported using immediate fact recall to calculate components in the mental computation tasks. It would appear that accurate and speedy recall of number facts would aid in mental computation, as more attention can be given to the overall calculations, rather than partial calculations. Sowder and Wheeler (1989) and Hope and Sherrill (1987) also posited this.

In contrast, unistrategy students who were as accurate as the accurate multistrategy users, but employed one strategy consistently throughout the mental computation (by definition), employed right to left separation. As argued in study 1, this strategy reflects the traditional pen and paper algorithm taught in Queensland schools.

These accurate unistrategy users were less proficient at computational estimation, and, although scored well on the number facts test, did not use immediate fact recall and derived facts strategies as often; that is their number facts strategies were not as advanced. It appears that multistrategy users, although no more accurate than those who were not flexible, were able to manipulate numbers with more understanding.

Strategy use

Considering that only 32 children were chosen for this study, and only 16 were chosen on the basis of using a variety of strategies, a substantial diversity of strategies was reported. Further, each question was answered in a variety of ways. However, the variety of subtraction strategies exceeded those for addition. It was posited that the difficulty children in this study had with subtraction (Fuson, 1992) resulted in their generating their own strategies for solving subtraction, more so than for addition.

With regard to strategy use, separation strategies were used more frequently than aggregation strategies, although less accurately. Beishuizen (1993) also reported this. A possible reason for the higher accuracy was that less load is placed on working memory when aggregating. Right to left separation was the most popular strategy of all, in particular with algorithmic exercises. However, it resulted in the most short term memory errors. Many children reported forgetting their partial answers, and having to start again when working right to left.

For subtraction word problems, the semantic structure was not reflected in the solution strategy; that is, both additive and subtractive strategies were used for both separate and missing addend subtraction, although subtractive strategies were more popular.

Summary and implications

Evidence from this study indicated that students had developed mental strategies (and also estimation strategies) without formal classroom instruction. Many students resorted to the right to left separation strategy, without first considering the numbers involved. It was argued this strategy reflects the school taught pen and paper algorithm, the teaching of which appears to have resulted in an overdependence by many students. However, there were some students who evidently looked at the numbers first, and made a decision regarding the appropriateness of particular strategies. These students who were both flexible and accurate were also proficient at both computational estimation and number fact knowledge, that is, these students exhibited a propensity for number understanding.

However, questions arose from this study. What allowed these students to be more inclined to access different strategies? Are other factors involved, for instance, affective factors? What qualities would younger children who had not become too dependent on pen and paper algorithms exhibit? Why are some children better mental computers than others? As a result of these questions the third study, to be reported here, was conceived.

Study 3

The aim of study 3 is to explain why some children are better at mental computation than others. The study is still in progress. The study was preceded by a pilot to develop instruments.

For the pilot study, sixteen year 3 children from one classroom in an inner city Brisbane school were interviewed, using mental computation tasks (similar to those used in study 2), to identify good mental computers. Because it was of interest to describe characteristics of not only accurate, but also flexible mental computers, children were selected on the basis of accuracy and flexibility. This paper will report on one child, Clare, who was accurate and flexible. Reference will be made to other students for comparison, particularly two other children who were also accurate: Emily who was also flexible like Clare, although there were differences as will be seen in the following discussion; and, by contrast, Mandy who was not flexible.

To achieve this, some possible aspects were identified from the literature in order to be able to commence the investigation. These were: number sense, particularly number facts, computational estimation, numeration, and properties of number and operation; social and affective issues including beliefs, values, and social context (e.g., classroom and home); and cognitive factors such as metacognitive processes and mental representations.

Connections between mental computation and other aspects

Skilled mental computers use a variety of strategies in different situations (depending on numbers and context), because they are disposed to making sense of mathematics (Hope, 1985; Maier, 1977; Sowder, 1994). Therefore, they must be aware of a variety of strategies. How do they choose which strategy to use? There is evidence of awareness of reflection and regulation. Reys, Bestgen, Rybolt, and Wyatt (1980), Hope (1987), Dowker (1990) reported children and adults choosing strategies based on their knowledge of number and operations, and choosing appropriate strategies to deal with the problems.

It is not sufficient to be aware of alternative strategies, but also to have the confidence to use them. The reasons that some children are unable to use better strategies than the pen and paper algorithms in different situations, vary. It may be because of prolonged practice of these algorithms, and/or being unaware of alternatives. It may also be because of a lack of confidence in experimentation and lack of belief in their own ability to choose more appropriate strategies, or lack of belief in appropriateness of using alternative strategies. Thus, the study of good mental computers may go beyond cognition and metacognition, to affects and beliefs (Sowder, 1994).

Connections have also been drawn between mental computation and other factors, including numeration and place value, number sense, computational estimation and number fact knowledge. Research has suggested that mental computation requires an understanding of numeration (Reys, 1985) and place value (McIntosh, 1996; Sowder, 1992). McIntosh (1996), Sowder (1992), and Trafton (1992) specifically mentioned mental computation as an indicator or element of number sense. It appears that mental computation and computational estimation may be related (Heirdsfield, 1996; Maier, 1977; Reys, Bestgen, Rybolt, & Wyatt, 1982; Sowder & Wheeler, 1989). Further, results of research (Hope & Sherrill, 1987; Sowder & Wheeler, 1989) identified basic fact knowledge as a related skill to mental computation. Mental computation has also been linked to number sense (McIntosh, 1996; McIntosh, Reys, &

Reys, 1992; Reys, 1984; Sowder, 1990, 1992). The ability to manipulate numbers appropriately in different contexts would facilitate flexible mental computation.

Plunkett (1979) suggested that mental algorithms are often iconic, for instance, incorporating the use of a number line, or number square. Reys (1985) stated that mental computation utilises visual thinking skills, for example, pictorial models. In recent years, some research has considered young children's mental representations of number (Thomas & Mulligan, 1995; Thomas, Mulligan, & Goldin, 1996, 1994) and how the development of children's representations can aid in the development of number (Bobis, 1993). However, in a study of young children's representation of the counting sequence 1 to 100 (Thomas, Mulligan, & Goldin, 1994), it was found that young children do not naturally view numbers in conventional ways (e.g., number lines, 99 board), but rather, in very idiosyncratic forms; although, in older children the number line and 99 or 100 chart began to appear (Thomas & Mulligan, 1995). Further, children with better developed number sense represented numbers in a dynamic mode; whereas, children with less developed number sense represented number in a static mode. This notion of dynamic imagery was also supported by Trafton (1992) when describing the metaphoric language used by students, for instance, "chop in half', "knock off', "tack on numbers". Here, students are assigning meaning to the symbols. It would appear that children's mental representations of number and operations may be factors in mental computation.

The interviews

While it is recognised that some of these aspects, described in the previous section, may be essential components of mental computation, others may not be as closely linked. With these aspects in mind, an investigative study of mental computers was initiated.

After the students were selected, they participated in a variety of indepth clinical interviews. After reviewing the videotaped interviews, it was often necessary to have the students involved in further interviews for clarification. Specific items addressing further mental computation (Table 2), number fact knowledge, computational estimation, number and numeration, and mental representations were presented. Other questions relating to self efficacy, beliefs, and metacognition were included in the interviews.

Mental computation, computational estimation, and number fact responses were analysed for strategy choice, flexibility, accuracy, understanding of number and numeration, and metacognition. Number and operations tasks were analysed for understanding of associativity and inverses, and relationships (e.g., 69-43=26, (69-44=25). Analysis of students' responses to numeration tasks were based on Ross's five levels (1986). Although analysis of individual interviews were undertaken separately, commonalities across interviews were considered, for instance, whether understanding of noncanonical partitioning of numbers was used for mental computation. In order to get a feel for classroom and home contexts, the children were encouraged to indulge in general conversation, and the teacher was invited to respond to initial and general inferences.

Table 2

Number combinations for mental computation word problems

Question type	Addition	Subtraction
basic fact	6+8	15-8
basic fact & ± 9	9+7	14-9
multiples of 10	64+20	76-20
2 digit w/o regrouping	53+34	58-36
2 digit with regrouping & including no. fact	46+28	65-28
2 digit, near compatibles	75+28	80-49
2 digit regroup, involving 9	45+19	63-29
bridge 100	76+43	107-15
3 digit, involving 9	246+199	234-99
3 digit, near compatibles	350+52	400-298

Clare's story

Clare was selected for further investigation as she was accurate and employed a variety of advanced strategies in the selection interview, e.g., 148+99: 100+99=199, 48-1=47, 247 (wholistic); 52-19: take 2 out of 9 = 7, 10-7=3, 4-1=30, 37 (this method was also reflected in the number facts test). She appeared confident in computation, and stated she liked mathematics, because she finds it easy and is therefore good, that is, she attributed her success to ability. This type of response was also elicited by Emily, another student who was both accurate and flexible ("I like maths, because it's my best subject."). This was in contrast to Mandy who attributed her success to practice. Mandy was also accurate in mental computation, but consistently employed a mental image of the pen and paper algorithm.

When asked how she knew she was correct, Clare replied, "I just think I'm right. I am usually right." In contrast, Emily and Mandy said they would check their answers by working through the examples the same way, and then wait for feedback from the teacher.

Clare stated that she believed she would be able to solve the mental computation questions, and she could. This was evident when asked at the beginning of some items, and also in a Student Preference Survey (SPS) (McIntosh, 1996). Both Emily and Mandy also stated they would be able to complete the tasks mentally, and could. Results of the SPS indicated that 5 of the 16 children believed they could complete all the examples on the survey, mentally. However, 3 or 4 of these children would not have been able to do so, as evidenced by their responses in the selection interviews, and when asked to solve some of the items on the survey. The three children already mentioned responded with a variety of "yes" and "no" replies to whether they would calculate mentally or not.

Clare attributed failure to "very foolish mistakes". Further, she needed to achieve, and only felt confident attempting questions if she believed she could succeed. After being unsuccessful at calculating 265-99 in the selection interview, she went home and asked her father how to calculate such examples. She was happy to attempt a similar question (234-99) in the next interview, because she now knew how to calculate it. However, she did not know why it worked ("That's what Dad told me to do."). Her confidence was also exhibited by her stating that her subtraction method (of levelling) "annoys Miss A...", but she was determined to continue to use it. However, she did realise that method was too complex for 3 digit examples. In the follow up mental computation interviews, when asked to think of another solution method, she saw no reason to think of a different method, except for the fun of it (appease the interviewer?). However, once she reasoned that some of her second methods were better than her first methods, she thought it was quite a good idea to indulge me. At times, hints had to be given, e.g., "what is 19 near?". Other times, no hints were given, for example, after solving 80-49 by 80-40= 40, take another 10, 10-9=1, 31, Clare then turned 49 into 50 and proceeded 80-50+1. Clare's confidence in her ability and her reluctance (at first) to try a different method was reflected across all her classroom work. She had a strong preference for her own methods, many of which she learnt from her father (although, not all the time, with understanding). Her later acceptance of alternative methods and even preference for these came as a shock to her teacher ("out of character for Clare"). It is suggested that she had nothing to prove to the interviewer by remaining adamant about the suitability or otherwise of alternative strategies.

Her ability to manipulate operations in this fashion was not consistent. In the number and operations interviews, she was not always sure whether to add or subtract one when taking away one more or one less (e.g., 73-45=28, 74-46=?). Thus, although her father had shown her a method based on this principle, there was little understanding. Emily, (also flexible and accurate) likewise had problems with this concept. However, she successfully used the idea in the mental computation interview without prompting. Her success in both mental computation and number and operations interviews was inconsistent. On the other hand, Mandy had to be deliberately encouraged to use strategies other than "calculating operations" (the term she used for pen and paper strategies). Mandy was successful at completing such tasks as: 257-100=157, so what does 257-99=? (with a fair amount of thought), but she stated that she still preferred "using operations". The three students had no problem with a similar concept for addition, that is, 234+99=333, because 234+100=334, and take 1, so 333. However, Mandy could not and would not use the concept for the mental computation tasks. In discussions with Mandy's teacher, it was revealed that Mandy had high expectations for accuracy and speed when completing tasks. This could explain her using the same "automatic" procedure for solutions, and maintaining confidence in this procedure. For all Clare's confidence, though, when asked to solve subtraction problems, she replied, "I don't particularly want to. I don't like doing take away in my head." This was despite the fact that she could. This attitude towards subtraction was reflected in her response in the SPS, where she responded positively to calculating mentally for only the simple subtraction problems. Thus, her preference for written calculation of subtraction was well founded on her knowledge of her poorer understanding of the operation. This negative attitude to subtraction was reflected in the class SPS responses. Four of the 16 students responded with "no" to all subtraction examples, and at least

3 others should have responded likewise, from indications in the mental computation selection interviews.

Clare admitted that she generally employed the first method "that pops into my head"; therefore, there were times she chose an arguably less efficient mental strategy. However, later in the interviews, such statements as, "why didn't I think of that in the first place?" indicated she began to consider strategy choice more carefully. Emily possessed a variety of strategies, but she admitted she also used the first method she thought of. In contrast to Clare, Emily did not show evidence of much regulation and monitoring, although she was encouraged to think of other strategies and decide which strategy she preferred. Mandy, on the other hand, had employed a mental image of the pen and paper algorithm in the selection interviews, and stated several times that she preferred that method and found it easier, as she was "used to it". Through prompting, Mandy developed a left to right aggregation strategy, and started to use it later in the interviews, because she said she wanted to practise the new way which may be easier for mental calculations. Mandy also was able to use a wholistic strategy for subtraction with 99, but stated, in all cases, she still preferred the "old way". In fact, when employing the new strategies, she still imagined the numbers one under the other, as though setting the examples out on paper.

Clare's number facts were fast and accurate. In the number facts test, she used recall (6 out of 16 times), and DFS (build to 10, pattern with 9, through 10 subtraction - like a levelling e.g., 17-9: take 7 out of 9 and out of 17, 10-2=8). The levelling strategy, as already mentioned, was used same strategy in the mental computation interviews for subtraction. Emily also used this levelling strategy in the number facts test and in the mental computation interviews. Both children stated they had not been taught this strategy, but worked it out for themselves. This offers support for children who employ derived facts strategies (DFS) understand relationships between numbers, and are able to use this understanding of number properties in mental computation. Emily also employed counting strategies in the number facts test. When it came to calculating in the mental computation interviews, counting made it difficult for her, as working memory was taken up with remembering counts, rather than attending to the calculation as a whole. One other child who used recall (not always accurately) and counting appeared to be so disadvantaged by her lack of number facts strategies, that the interviewer gave her answers to number facts so that she could complete the mental computation tasks. Mandy also used counting in both the number facts test and the mental computation interviews, but did not have the same memory overload problems. Most of the strategies Clare employed in the number facts test were reflected in the mental computation interviews. Her agility with number facts was an advantage in the mental computation interviews, as working memory was available for efficiently solving more complex problems. The children's teacher was amazed that the children had formulated such strategies. She stated that she had used similar strategies when modelling addition tasks, but did not expect the children to be able to use them for addition, and in particular, subtraction. It appears that Clare and Emily had the capacity to build up a rich, interconnected network of knowledge, and access this knowledge, readily.

Before the indepth mental computation interviews, the children were presented with the number facts test, in which Clare calculated 15-8 by levelling (quite a favourite take away method for her). She was the able to recall this fact for the same question in the mental computation interview, that is, she had learnt from the experience.

This was not the case with the either Emily or Mandy. They recalculated the answers to the number facts, although they had already done so, not 5 minutes before, for instance, use doubles, through 10.

Clare agreed that knowing number facts was important, but didn't know why, except that her teacher had told her. Emily stated that the importance of knowing her number facts was to be able to get them correct in daily tests. Mandy could see a benefit in the future, as they may be useful in a future profession, for instance, a scientist would need number facts. She also believed it was necessary to know them in order to be able to pass pen and paper tests. These responses surprised the teacher, as she had often used more worthwhile explanations for the need for immediate fact recall, for instance, ease of computation.

Computational estimation is poorly treated in the mathematics curriculum. Clare defined estimation as a "type of guessing", a definition in common with other children in her class. She stated that she only estimated when given classroom estimation tasks that were treated as rounding only. However, Clare did not employ rounding in the interview. Rather, she used other strategies more appropriate to the situations, for instance, truncation and wholistic. Because

Clare's mental computation was so good, she attempted to calculate accurately. This has been reported elsewhere (Heirdsfield, 1996; LeFevre, Greenham, & Waheed, 1993). It was decided to present Clare with additional 3 digit estimation questions that were too difficult for exact calculation. Clare's responses reflected an understanding of magnitude of number, place value, and the effect of operations. One example of a successfully completed task was: "Your friend has $152 and spends $144 on a cassette recorder. You have $156 and spend $142 on another cassette recorder. Who has more money left?" Response: "I do, because I started with more and spent less." Emily completed the computational estimation tasks using similar strategies as Clare. She also exhibited an understanding of the size of numbers, place value, and the effect of operations. However, the number combinations did not have to be altered to prevent her from calculating accurately. In contrast, Mandy could only relate estimation to measurement, and was generally unsuccessful at the estimation tasks.

The numeration tasks revealed Clare's understanding of both canonical and noncanonical representations of number (Ross, 1986). She was particularly flexible with different representations of such numbers as 560 (5x100 + 6x10 + 0x1; 56x10 + 0x1; 500x1 + 6x10; 55x10 + 10x1; 5x100 + 3x10 + 30x1) and 209 (2x100 + 0x10 + 9x1; 20x10 + 9x1; 209x1; 19x10 + 19x1). Although MAB (Multibase Arithmetic Blocks) were available, Clare did not use them. However, there were times the interviewer had to encourage her to elicit more combinations, although she appeared to delight in the challenge. In contrast, Mandy was slow at representing numbers in different ways. She had to be prompted with such questions as, "What about some ones?", and needed the support of MAB for many examples. Even with MAB, she did not show a solid understanding of what she was doing, as she constantly checked and recounted her manipulations. An alternative explanation could be that her need for absolute certainty overshadowed her understanding of number. However, it appears curious that she would have to count and recount tens to ones, if she truly understood regrouping. Emily also required MAB to represent alternatives, but she appeared to understand better what she was doing, as she manipulated the blocks faster and with more confidence.

Throughout the interviews, Clare, Emily and Mandy were asked whether they saw anything in their heads while calculating, estimating, and so on. A very definite "no" was the reply from each child. To investigate her mental representation of number, they were asked to close their eyes, think of the numbers between 1 and 100, and then put on paper what they saw in their heads (Thomas & Mulligan, 1995). Clare's drawing showed the numbers 1, 2, 44, 99, 100 (possibly from the rhyme, "1, 2, skip a few, 44, skip some more, 100"). The numbers 1, 2 and 100 were drawn with hands, and 44 and 99 with wings. She revealed that all double digit numbers would have wings. Clare also revealed that she didn't normally think of numbers in that way, but wanted to make them look interesting. Further, the numbers were not doing anything (not moving), but they were in order. Emily wrote the numbers 1 to 10 on one line, 11 to 20, on the next, 21 to 30 on the next, and so on, indicating some knowledge of structure of the number sequence. Further, she indicated that the numbers go across her forehead. She said she did not use a 99 or 100 board in class. Mandy's drawing showed the numbers 1, 2, and 3 on a circle. In trying to explain her drawing, she drew arrows from 1 to 2, 2 to 3, and 3 back to 1. Then she motioned with her hand that the numbers keep turning as of in a series of loops. Although she stated the all the numbers are involved, she only saw the numbers 1, 2, and 3 "going round and round".

During the course of interviews, Clare revealed things about her thinking, unprompted, for instance, "No, that can't be right", "I'm lost now", "I'm usually right", "This one's difficult", "This one's easier", "I like this one, because it has something to do with 99", and "Seventy-five is easier to use than 76, so I'll use 75". These statements revealed the existence of metacognitive processes and beliefs. Clare had access to a variety of strategies, but rarely consciously chose the most appropriate strategy for the number context. However, when encouraged to think of other strategies, she made judgements regarding the suitability of the strategies. Clare was confident in experimenting with different strategies. She seemed to disregard what was taught in the classroom, rarely using the taught algorithm to solve the problems mentally. In fact, Clare revealed that she often used her levelling strategy for subtraction to solve written exercises.

Concluding comments

For generations primary mathematics has focused on the teaching of algorithmic procedures, using one inflexible procedure for each operation. Changes that have occurred (e.g., the shift from the 'borrow and pay back' to the decomposition subtraction algorithm) have been in replacing one inflexibility with another. This has interpreted computation in simplistic terms and assumed children are programmable computers that can receive and reproduce fixed sequences of procedures. Left to their own devices, children use a variety of procedures depending on need, context and number size. Children see computational situations from a variety of perspectives, for example, some children see 7-3 as taking 3 from 7 and some as building 3 up to 7. This complexity is reflected in the real world situations that can be represented computationally. Hence, as the needs for mathematics turn from accuracy in computation (now the province of calculators and computers) to interpreting real-world problem situations, the inflexible 'do-it-one-way' traditional algorithms become a liability. Children need the flexibility that comes from constructing their own procedures for computation that is mental, recorded and estimated. And as the research above is showing, continuing foci on familiarity with fixed traditional algorithms is crushing such flexibility

The transition in teaching from inflexible pen and paper algorithms to self constructed mental procedures is a large step for teachers. In the first, fixed methods could be applied to all students in a similar manner. In the second, each student is a special individual case to be nurtured. Teachers need a repertoire of procedures, teaching techniques and diagnostic tools.

The three studies have moved from studying the existing situation in schools to looking at the relationship between mental computation and other number sense proficiencies. As the complex interaction between knowledge, affect and proficiency emerges, insight will hopefully also emerge in how to encourage students to be flexible and creative interpreters of their world from a computational perspective.

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