CHILDREN'S INFORMAL WRITTEN COMPUTATION METHODS

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Abstract

Teachers are being encouraged by recent curriculum documents to give greater emphasis to students' informal written computation methods. This raises the question of what is the nature of these informal methods, the extent to which students can and do use them, and how they might be best utilised in the teaching of mathematics. The present study explored the informal methods students in two middle grade classes of the primary school used to solve problems for which they had not previously been taught a formal method. The results of the study indicate that, once students can understand the problem, they invariably are able to generate a valid method to solve it. Furthermore, while the sophistication of their methods varied considerably, the students displayed an ownership and understanding of the methods which they do not always have when using formal algorithms. Invalid methods were nearly always associated with attempts to obtain an answer just by symbolic manipulation of the numbers involved. This has important implications for the teaching of written computation in mathematics. In many cases, it may not be necessary to teach the standard algorithms at all and, if they are needed, the students' informal methods could form a valuable basis for this teaching.

Influential curriculum documents such as A National Statement of Mathematics for Australian Schools (Australian Education Council, 1990) and the Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989) have advocated a number of quite significant changes in mathematics education. Important among these proposed changes has been the recommendation for teaching approaches to encourage students to develop and use their own informal methods for solving mathematical tasks. For example, the Australian document (Australian Education Council, 1990) notes that 'children should be encouraged to explore and invent informal paper-and-pencil methods' (p. 111) and, further, 'they should devise a variety of paper-and-pencil computational strategies, record the results of their thinking in their own way, and progress towards efficient, although not necessarily standard, procedures' (p. 117).

This encouragement of children's own strategies stands in contrast to the more common approach of the children being formally taught an established routine or algorithm for their written computations.

The recommended emphasis on children developing and using their own informal methods appears to be based on a number of factors. One of these is that the students' understanding and achievement with the formal methods they are taught in the classroom are often poor (see, for example, Willis, 1990; and Ginsburg & Allardice, 1984). The following example from Jessica, a Grade 5 student, highlights the type of errors often made when children do not fully grasp ideas involved in an algorithm:

(Jessica, Gr 5): 23
x14

92
230
1150

In talking with her teacher afterwards, Jessica said, "I thought I'd give it a go. I put the nought down and something went wrong". And it was clear that she did not know what it was that 'went wrong'.

However, while children may not be successful with the written algorithms, it is not necessarily a result of them not understanding the mathematics. In a study with Grade 3 children involving multidigit computation, Carraher, Carraher & Schliemann (1987) found that the children were very successful (80-90% correct) when they solved the problems in real contexts but that this fell to only 20-30% when they used the school-taught algorithms. Despite careful teaching of these processes, when children are working with the algorithms they can focus too closely on manipulating the symbols and so suspend thinking. Benezet (1935) had suggested some years ago that the effect of this teaching had been 'to dull and almost chloroform the child's reasoning faculties' (p.242) while Stein (1988), paraphrasing Gresham's Law in economics, put it as 'algorithm drives out thought' (p.79).

A second factor influencing this changed emphasis in computation is that children can and do develop their own methods for solving problems and that they use these with confidence and success (see, for example, Suggate, 1995; Kamii, Lewis & Livingston, 1993; Mulligan, 1992; Carraher, 1988; and Carpenter & Moser, 1984). Recent research by HedrÈn (1996, 1995) with classes in Year 2 and Years 4 and 5, has demonstrated the diversity of the successful paper and pencil strategies children use when they are given the opportunity to think about the solutions in their own way. Similarly, children use a range of valid, individual methods when working with mental computation (Thornton, Jones & Neal, 1995; Sowder, 1990; and Hope & Sherrill, 1987). While the thinking is similar in each case, we tend not to see as much evidence of informal written methods in the classroom because there is usually firmer direction given about the way children should record their work on paper. However, as HedrÈn notes, this informal written computation is a kind of 'written mental computation' in which the writing is just 'some rough notes to support your memory' (1996, p.1). Importantly, whether they be mental or written, children's informal methods lead to fewer mistakes. They are the children's own methods and so will not involve anything they do not understand nor are they likely to be used in a rote, unthinking way.

The significance of children's own informal methods is further underlined in the extensive literature on the constructivist perspective of learning. A fundamental position of constructivism is that knowledge is constructed by the individual as a result of the learner's personal activity and that it cannot be acquired passively from someone else (see, for example, Simon, 1995; Post, 1992 & von Glasersfeld, 1991). In this way, as Post (1992) has observed, learning becomes 'an intensely personal affair' (p.17) and so the methods produced by the children themselves are of fundamental importance in the learning process.

Children's strategies may be seen as intuitive, natural approaches or they can be quite structured. They are 'informal' methods in that they are not the methods taught formally in the classroom even though at times they might resemble them. Whatever the form of these strategies, Colomb, Kov·cs, Lluis, Rathmell, Shuard & Streefland (1988) and others have suggested that they are at the core of the teaching and learning process. The important question raised by this group is how can we best deal with the informal methods in our teaching and what teaching methods are compatible with the development of the children's own strategies. To answer this, we need to have some idea of the nature of the informal methods children do use and to investigate the ways in which they might relate to the methods we would normally teach them.

It was the purpose of the present study to examine this for the case of written computation methods.

The first topic selected for the investigation was that of multiplication involving two 2-digit numbers, the normal algorithm for which is 'long multiplication'. An additional topic involving area of a rectangle but in a non-standard format was included to provide an alternative setting. A study of multiplication (and division) with smaller numbers had been undertaken by Mulligan (1992) in which she conducted a two-year longitudinal study with a group of children initially in Year 2. She investigated the children's solution strategies for multiplication and division word problems. This extended an earlier study by Carpenter and Moser (1984) with addition and subtraction. Mulligan found that 75% of the children were able to solve the problems although they had not received formal instruction in multiplication or division during most of the period of the study. Also, a wide variety of strategies were used. Mulligan classified these using a scheme which integrated the 'level of abstractness' and the 'level of modelling'(p.33). Three basic levels of strategy were identified:

i) direct modelling and counting, using counters or fingers;
ii) no direct modelling but with counting, addition and subtraction

strategies; and
iii) use of known or derived addition and multiplication facts.
Mulligan noted that this range of strategies was analogous to that observed by Carpenter and Moser (1984) for addition and subtraction, although the strategies were somewhat more complex and diverse. This suggests a similar classification framework may be appropriate for strategies used with written computation tasks involving larger numbers.

The aim of the present study was to examine the nature of the informal methods children use for multiplication with two 2-digit numbers. It also endeavoured to examine the strategies used in a novel area problem in which the children had to determine the number of squares of a specified size which would fit into a given rectangle. As this is not a common problem, there is no established algorithm taught. However, an algorithm could be readily developed based on the familiar procedure of A = L x W. It was of interest to see if there were any parallels in the nature of the approaches used in the two tasks. Finally, for the multiplication task in particular, the study hoped to determine in what ways, if at all, the informal methods relate to the formal algorithm normally taught in schools and whether any prior teaching of an algorithm influenced the methods used.

Method

Subjects

The subjects were 38 children from two composite classes in the middle grades. They were from two neighbouring private schools located in a regional centre of Victoria, Australia. The first class comprised third and fourth grade children while the second consisted of children in grades three and five. The Gr 3/4 class had 23 children and the Gr 3/5 class had 22, although only 18 were present for the study. There was a policy of 'mixed age' or 'composite' grades in both schools. For most of the normal teaching, no distinction was made in the content presented to the children from the different grade levels in the same class although there would be provision for children to be able to work at levels appropriate to their abilities. Both of the teachers were creative and enthusiastic about their teaching responsibilities; they handled their respective composite groupings very well. Prior to the study, the Grade 3/4 class had not received any instruction on 2-digit multiplication although the Grade 4 children in the class had been taught 1-digit by 2-digit multiplication the previous year. The teacher in the Grade 3/5 class had introduced 2-digit multiplication to the children in both grades approximately two months earlier but had not covered it since that time. Children from neither class had been taught the rule for the area of a rectangle although, even if they had, it would not have been directly applicable to the task presented.

Procedure

Both classes were given the same problem of 23 x 14 to solve. It was presented by their normal class teacher using the same context. Bags with 14 'Smarties' (M & Ms) in each were prepared prior to the lesson and displayed at the front of the class. The task set the children was to determine the total number of Smarties that had been needed to produce a bag for everyone in the class, a total of 23 students (or 22 students and the teacher). They were able to use any method they chose but were asked to do all of their working on the paper provided and to write down a story of how they had done their work. On completion of the task, with their solution set out on the page, the children were given their bag of Smarties.

The area question was presented orally in the context of making a place mat for the table using the question: 'How many coloured paper squares 5cm x 5cm are needed to make a mat 30cm x 45 cm?'. It was introduced through a discussion of the need for place mats for the dinner table, the shape of the mats (rectangular) and the dimensions of the mats (45 cm long and 30 cm wide). The children were told they could use any method they wished and were asked to write down how they worked it out on the sheet provided. The nature of the area task was different from the multiplication problem in a number of ways. Firstly, the actual arithmetic involved was considerably easier but the introduction of the notion of area presented an additional conceptual load. Finally, there was a need for multiplicative reasoning related to the five rows of nine squares if the children were to use methods other than counting by ones. Because of time restraints, only the Grade 3/4 class completed this task.

Analysis of the results was to be qualitative. While some statistical analysis of the results from the two classes on the multiplication task could have been done, this was thought not to serve any real purpose. The major interest was the nature of the responses rather than a statistical comparison of results between the two classes. Similarly, the correctness of the answer was, by itself, not seen to be of importance in the analysis. It was the validity of the method which was considered. However, incorrect answers generated by flawed strategies were noted.

The same basic classification scheme devised by Mulligan (1992) was used in the study to categorise the methods the children used. However, sub-levels were added to accommodate the range of responses produced:

Level 1: Direct modelling and counting
Level 1A - counting by 1s
Level 1B - counting by 10s and 1s or other combinations
Level 2: No direct modelling but use of counting and/or addition strategies
Level 2A - group counting
Level 2B - group counting using 10s
Level 3: Use of number facts and properties
Level 3A - use of known or derived facts
Level 3B - use of distributive property and other laws of arithmetic

At Level 1, the actual situation was drawn and the result found by counting, usually by ones or by tens and ones. This distinction between the strategies was noted the in the two sub-levels, A and B. For Level 2, the strategies were similar but no diagrams were employed.

Symbols were used to record the process, such as counting in groups of 14. Level 3 strategies were also classified under two sub-levels. Strategies drawing on known or derived number facts were classed as Level 3A while those drawing on such things as the distributive property were classed as Level 3B. These three levels covered what might be called the valid, informal strategies. In addition, some students used strategies which were not valid because of some conceptual error and others used a version of the standard formal algorithm. These were noted separately from the strategies classified under the above scheme as were the situations where a student was unable to make any meaningful attempt at the task.

Results

For the multiplication of the two 2-digit numbers (23x14), counting strategies dominated with nearly half of the children across the two classes forming 23 groups of 14 and counting them (see Table 1). This was more pronounced in the Gr 3/4 class with 57% of the students using the model/counting method. None of the students in Gr 3/4 counted by 10s although three children in the Gr 3/5 class utilised this approach. The next most popular method was to add on by 14s, usually done mentally by adding 10 then 4, or by recording the 23 14s and adding them using normal column addition. This method was used by eight of the Gr 3/5 class and three students in the Gr 3/4 class. It was clearly the most popular strategy for the Gr 3/5 group (44%). It would appear that they were more familiar with the addition of 2-digit numbers than children in the other class. Three students in each class made use of the distributive property by breaking the numbers up and multiplying 14x20 and 14x3 or 10x23 and 4x23. A variation of this was to form 14x10, then double it and add on 14x3. Given the relatively small number of students who used this property (15%), it would seem that it is not something with which children in the middle grades are generally familiar and do not have as part of their mathematical understandings.

Table 1
Classification of Children's Methods for 2-digit by 2-digit Numbers
(Q: 14x23)

* 10 incorrect results, all due to counting errors.
**Gr 3/4:Three students attempted to use a formal, symbolic method in

that they appeared to have tried to adapt a combination of the algorithm for addition and that for the multiplication of 1-digit by 2-digit numbers by multiplying ones by ones and then tens by tens. They did not subsequently use an informal method

Gr 3/5:Six students initially tried to use the standard algorithm but were not able to apply it correctly. They then successfully used an informal method and have been listed under the appropriate category for their respective methods.

A total of eleven students across the two classes attempted to use what could be considered to be a formal method. Only two of these did so correctly. Three students in the Gr 3/4 class appeared to have tried to apply other algorithms with which they were familiar and multiplied the ones by the ones (4x3) then the tens by the tens (10x20). The example below from Riki illustrates the way these children tried to handle the question:

(Riki, Gr 4) 23
x 114

42

"Four 3s are 12; put down the two and carry the one. One and one are two, times two, is four."

One student in each class successfully used the standard algorithm of 'long multiplication' while another six children in the Gr 3/5 class attempted to do so. Only the students in Gr 3/5 had been taught the algorithm so presumably the solitary Gr 3/4 student had been taught the process by an enthusiastic family member. It is of interest that only a total of seven of the 18 Gr 3/5 children attempted to use the algorithm they had been taught some two months earlier and, of these, only one obtained a correct solution. All were successful, however, when they adopted an informal method of their own. The comments of two of these students to their teacher reveal that they were keen to use the algorithm but were reasonably happy to use their 'fall-back' strategy when it did not work even though it took much more time.

Adrienne (Gr 3):"I started doing a multiplication but I couldn't do it properly." (She then drew 23 lines of 14 small circles and counted by ones.) "I wanted them to look like smarties. It takes a very long time to do it - and to remember how many you've done."

Laura (Gr 5):"Other things (the formal algorithm) I did weren't right. I started timesing it by 14. I'm not sure why that didn't work. I chose this way at the end when I couldn't do it."

Final method used: 14 28 42... ...308
+14+14+14+14
28 42 56 322

The results from the area task are given in Table 2. Over half (55%) of the children used a diagram with each of the 54 squares marked onto the rectangular representation of the 30cm x 45cm table mat. One student drew a full-size rectangle on a large sheet of project paper but most who drew diagrams used scaled down versions of the situation. None of the children used a counting technique without having a model of the squares on the rectangle. This is perhaps not surprising given the nature of the task. Five of the children used methods which were classified at Levels 3A or 3B. These methods were based on the deduction that six 5cm squares would fit along the width of the rectangle and nine along the length to give a total of 9x6 squares altogether. This appears to be a relatively sophisticated approach for children at this level and reflects an ability to draw on a number of spatial and numerical concepts and understandings to solve the task. It seems the difficulty for the four students who were unable to generate a satisfactory method at all for the task was that they were unable to visualise the paper squares on the rectangular mat. Most then tried to manipulate the numbers in some way to produce an answer. For example, one response was "I worked it out by you add up the sides 45+45=90 and 30+30=60 so 90+60=150 so 150 is your answer". In general, this task proved to be more difficult than the multiplication task even though the actual computation involved was relatively simple. The spatial and measurement ideas which were needed presented considerable difficulties for a number of the children.

Table 2
Classification of Children's Methods for Area Task
(Q: How many squares 5cmx5cm are needed to cover a table mat 30cm x 45cm?)

Method	Level	Gr 3/4(n = 20)	(%)
Models; counting	1A	7	
	1B	4	(55)
Addition strategies; no modelling			
	2A	0	
	2B	0	(0)
Derived facts; number properties			
	3A	3	
	3B	2	(25)
Invalid method/no response		4	(20)

Discussion

The results from the study indicate that, once students can understand the problem, they invariably are able to generate a valid method to solve it. Invalid methods were nearly always associated with attempts to obtain an answer just by manipulation of the symbols involved. However, while all of the informal methods the children used enabled them to obtain solutions, the sophistication of their methods varied considerably. The modified classification scheme based on the system used by Mulligan (1992) proved to be quite an effective means in identifying the different levels among the children's responses. It is perhaps not surprising that the most common methods among these middle-grade children were at Level 1. Representing the task in a model and using counting techniques, often by ones, is consistent with the concrete, counting-based knowledge of mathematics these children have (Baroody, 1989). Some of the children were able to make use of more complex mathematical understandings in the methods they followed. Although this was a relatively small proportion (approximately 15%), the use of such ideas as the distributive property in the Level 3 methods does suggest that children at this stage are able to draw together relevant knowledge to produce their own efficient methods. These methods clearly resemble the more standard algorithms but the major difference is that the methods are the children's 'own' and so they do not need to rely on a rote, step-by-step process to use them. It would be of interest to examine whether, over an extended time, more children made use of these ideas as their experiences continued to broaden and if they were encouraged to explore their own informal methods for solving computation tasks.

Following a school-taught procedure does not necessarily mean that the children are doing it in an unthinking manner or without understanding. However, the results with the class which had been taught the algorithm for 2-digit multiplication suggest that, in general, the children did not understand the process. This result is consistent with previous findings about children's use of the formal algorithms (Carraher, 1988, and others). It was of interest that, indeed, most of the children in the class (11 of 18) avoided altogether trying to use the standard method they had been previously taught. They preferred, instead, to use their own methods, while of the seven students who did use the algorithm, only one was successful. Clearly, though, this result needs to be viewed with some caution. Only one class was involved and it had been two months since the algorithm had been presented to the children. Nevertheless, the class teacher is outstanding and he believed he had taught it well. Understandably, he was a little disappointed with the outcome but it only emphasises the observation of Clarke & Clarke (1987) that 'meaningful, long-term learning will occur to the extent that the procedures and results of a student's application of formal mathematics have intuitive validity' (p.127). The children had not yet made the connection between the formal mathematics of the algorithm and their relatively concrete and informal personal knowledge.

Children are good at producing computation methods appropriate to their own level of knowledge and understanding. Furthermore, they can be used as the basis of much of the teaching in number work to form the 'backbone of the teaching/learning process' (Colomb, et al., 1988, p. 124). I do not believe that one should be disheartened by the frequent use of counting-based strategies identified in this study. It is not a reason to disregard the children's informal methods but, rather, be seen as an indication of the appropriate starting points for the teaching of any more formal procedures. That some of the children produced quite sophisticated and efficient informal methods suggest also that it might not be necessary to teach them the standard, formal process at all.

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