Patterns and Relationships: Conceptual Development in Mathematical (small scale) and Real (large scale) Space

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ABSTRACT

The everyday ways of knowing that children bring to the classroom are becoming increasingly interesting to researchers wanting to understand meanings associated with problems in mathematics and geographic education. Patterns and relationships identified in visually presented problems have been shown to be embedded in children's prior knowledge and experience.

In the research reported this proposition is explored in terms of the correspondence between ways of presenting problems and the strategies used for solutions to problems in small scale space (Euclidean) and photographs of large scale 'real' space. Samples of grade seven and grade ten (N=86) students were selected from two urban high schools with contrasting neighbourhood characteristics.

Some gender differences and school and neighbourhood specific differences were recorded in the data. Discourse analysis suggests that both may be explained, in part, by contextual factors related to home,
family and location.

INTRODUCTION
Problem solving strategies used by children when processing visually presented stimuli pose questions related to prior knowledge and understanding, cognitive strategies and beliefs about the nature of intelligent behaviour. Yet the literature extant has few references to children's learning in spatial/visual domains despite their increasingly visual and computerised learning environments. In mathematics problem solving, for instance, findings by Shama and Dreyfus (1994) and Taplin and Robertson (1995) suggest that equally successful outcomes can be produced using either visual or algebraic strategies. If this research can be verified then there are many questions raised for determining relevant teaching and learning approaches in mathematics education (Collis, Watson and Campbell, 1993). One way is to look for guidance from other disciplines. For instance in geographic education there is quite a lengthy research history related to visual imagery (Downs and Stea, 1973). Concept or mental mapping as a cognitive construct has been studied by geographers and psychologists since Tolman's seminal paper of 1948. And, in a related field the study of environmental perceptions (Robertson, 1994; Walmsley, 1988) have provided insights to the development of mental constructs as diverse as language, number, spatial distribution and density. The question raised is whether Euclidean and real world space rely on similar internal processes. Some cross cultural research suggests this may be the case (Bishop, 1983; Sternberg, 1987). If this is so then there is value in looking to ways in which children problem solve in 'real' space as well as mathematical or Euclidean space.

Expressing generality from patterns is a notion that is fundamental to the development of mathematical and geographic concepts. Rules that relate to generalities help simplify mathematical relationships. At the same time recognition of order or arrangements of phenomena, both natural and social in the real world help to simplify daily interactions with the environment. In the natural environment, for instance, patterns of 'mathematical' symmetry have often been observed and remind us that the language of mathematics may have its inspiration in the order of life around us. And, not surprisingly, in the social and built environments diverse communication networks give order to spatial movements in global, national and regional movements of people and commodities. Pattern recognition is an important component of the National Statement on Mathematics for Australian Schools and directly relates to sections including Place and Space, Systems, Time and Change in the National Statement on Studies of Society and Environment.
The search for patterns and relationships in visually presented problems is a personal problem. Cognitive representations of problems presented are not open to teachers and researchers to interact with or observe. Nevertheless, they provide a way of testing the hypothesis that similar strategies can be used when representing visual-spatial problems in small scale space and real space. The challenge for the researcher is to find reliable strategies that enable us to access the meanings of learners. One technique being widely applied is the reconstruction of meanings through children's recalled imagery (Bishop, 1991; Werner, 1991). While this may inform our queries related to specific problems the sources of understandings may not be so clear. Hence, part of this process may be to consider more closely children's way of knowing (Biggs and Collis, 1991) and how they interact with their environment through leisure and recreation activities or learning informally from home and neighbourhood as well as the formal school learning (Robertson, 1995a; 1995b).

Specifically, the aim of the research described is to develop these arguments further and seek knowledge of any mismatches between teachers' mode of delivery and the approaches which students are most likely to apply to the task. The rationale has drawn on current research in educational psychology, developmental psychology, mathematics and geographic education. The central concern that synthesises this review is an interest in children's understandings and meanings of patterns in space - both Euclidean and real.

THE CONCEPTUAL FRAMEWORK

The conceptual framework for the research discussed is summarised in Figure 1.

Figure 1: Conceptual overview of project

Apart from issues of prior knowledge and experience related to visual/spatial reasoning and the impact of informal as well as formal learning, recognition is given to the importance of investigating children's responses and their preferred ways of processing. In fact the format presentation of a problem may relate to children's background in ways that enhance or limit the response. One way of verifying this either way is so assess the quality of the response given and make comparisons between responses to a range of problems presented in different formats. That way there is a chance to verify the personal preference for a particular problem presentation. The use of SOLO
taxonomy enables this assessment of response outcome (Biggs and Collis, 1991).

A pilot study conducted in 1994 provided encouraging support for the conceptual framework and some of the hypothesised relationships (Robertson and Taplin, 1994; Taplin and Robertson, 1995). The pilot study focused on mathematical problem solving with particular reference to the expression of generalisations from numerical and spatial patterning tasks. Part of the analysis explored links between students' pattern formation and generalisation and the SOLO Taxonomy (Biggs and Collis, 1991). The results, which were based on interview data collected from grade seven students (n=40) in two contrasting socio-economic areas, provided some support for the hypothesis that students have preferred methods of problem solving and that these do not necessarily relate to the ways in which the information is represented. Along with some gender differences in the data there were indications also of shared preferences related to school location, possibly associated with prior knowledge or 'everyday' intelligence (Biggs and Collis, 1991). There was also a suggestion, which warrants further investigation, of links to the SOLO Taxonomy unistructural-multistructural-relational cycle (Biggs and Collis, 1991) in students' attempts to recognise generalisations from the patterns they formed (Taplin and Robertson, 1995).

Links to large scale or real space were not considered in the pilot study. In 1995 the research has been taken further to include problem solving related to patterns and relationships in small scale and large scale real space and with two age cohorts in the same school settings as those used in the pilot study.

THE 1995 STUDY AND FINDINGS

Based on the outcomes of the 1994 pilot study the 1995 study considered the following research questions in relation to visual-spatial patterning tasks both from small scale and large scale space.

1. Is the predominant preference for concrete modelling reported by Robertson and Taplin (1994) a phase in the student's development?

2. Is there a preferred problem solving approach which is used consistently over time?

3. Is there a match between students' problem solving with small scale and large scale contexts?

4. Are there gender differences in problem solving approaches?

5. Is there an observable progression in students' ability to recognise
generalisations from their representations of spatial patterns?

6. What kinds of external representations do students make of patterns, and what intuitive or concrete symbolic processes do they apply to these representations, in order to form generalisations?

Methods and techniques

Sampling:
The same two schools used in the 1994 study were used for data collection in the 1995 study. School A is located in a low-income urban fringe neighbourhood with high levels of public housing, unemployment and complex family structures. School B is located in a middle income urban neighbourhood where employment and family structure appear to be much more stable. In each school, random samples of gender balanced groups were selected from each of Grades 7 and 10 (N=86). Using a second Grade 7 cohort was intended to provide longitudinal data for comparisons with the results of the 1994 study.

Tasks:
Problem solving tasks required students to express generalisations from patterns that are typical of patterning tasks being used in schools.

The ways in which tasks relate to Euclidean or small scale space and 'real world' space are described below.

1. Tasks related to Euclidean or small scale space
Two patterns selected for tasks were specifically mathematical and lent themselves to the expression of algebraic generalisations from the patterns. They were a 'path' problem and a 'step' problem (see Figure 2). Both were used in the pilot study. To allow for students' preferences, two formats were selected in which to present these patterns. Specifically:
• concrete modelling, in which a representation of the pattern is made from blocks or other materials.
• a diagrammatic representation.

Figure 2: Small scale space pattern

(a) The Path Problem

(b) The Step Problem
2. Tasks related to 'real world' space

A further two patterns related to 'real' world or large scale space. In order to simulate tasks similar to those described for mathematical space, geographic locations were selected that have similar patterning relationships in land use distribution. Using aerial photographs and sketches (see Figures 3), students were required to assess land rent principles of sites identified in proximity to small urban communities. The pattern represented refers to the Land-Rent principle of declining land rents and financial returns with increasing distance from a commercial centre (Haggett, 1979).

Methodology
Data collection involved individual, semi-structured interviews, each of approximately 20 minutes duration. Observation and Teachback (Pask, 1976) strategies were used to monitor the students' responses.

For Tasks 1 and 2 related to 'small scale' - Euclidean or mathematical space (ie path and step problems respectively) students were given a sequence of tasks related to pattern identification. They were shown the first three patterns as indicated in Figure 2 and then asked what the fifth, tenth and hundredth steps would look like and finally asked if they recognised a rule that would apply to any step. In order to ensure that the format of the representation of each pattern did not influence the student's mode of response, these were be presented in a cyclic rotation of concrete and diagrammatic formats of the same problem.

For tasks related to 'real world' space the central problem of seeking a pattern from the information and then attempting to generalise this finding remained the same. The question relating to Figure 3 was

If the Jones live at location \(\text{at}\) and their rates are $1200 per year
AND, the Smiths live at location 2 and their rates are $1000 per year,
AND, the Blacks live at location 3 and their rates are $800
Where would you expect the Thompsons if their rates are $600 ? Why?

Students were then asked to suggest a rule that would explain the differences and help decide what the rates will be at any location. Finally, they were asked to represent the rule with a diagram or graph.

RESULTS

Small Scale Tasks
Although data were analysed using Stat-View 5 few statistical comparisons are possible as the individual data cells are too small. As intended by the researchers all this points to Figure 3: Photographic and photosketch landscape representations

the need for a much more extended sample. Nevertheless preliminary investigations using frequency tables and chi-square analyses do provide some interesting findings. The data have been analysed for two views. First, to determine differences that may reflect cognitive development with age and/or gender. Second, to see if there are any school specific differences.

Whole Grade View

Table 1 reports summary data for both grade groups for small scale Tasks 1, the Step Problem and Task 2, the Path Problem. The data indicate an increase in verbal responses in Grade 10 with statistically significant differences between grades for Task 2 (p<.05) However, it is notable that regardless of task by far the preferred strategy for both grades was in concrete format. Furthermore, when broken down by gender the data showed no significant differences with both boys and girls indicating a strong preference for the concrete format. This is a departure from the view gained in the previous year’s pilot study cohort (Robertson and Taplin, 1994).

Table 1: Summary data for Tasks 1 and 2 by Grade

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Grade 7</th>
<th>Grade 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete</td>
<td>24 53</td>
<td>16 39</td>
</tr>
<tr>
<td>Diagram</td>
<td>4 9</td>
<td>2 5</td>
</tr>
<tr>
<td>Mixed Mode</td>
<td>9 20</td>
<td>6 15</td>
</tr>
<tr>
<td>Verbal</td>
<td>8 18</td>
<td>17 41</td>
</tr>
<tr>
<td></td>
<td>n=45 100%</td>
<td>n=41 100%</td>
</tr>
<tr>
<td>Strategies</td>
<td>Task 2</td>
<td>Task 2</td>
</tr>
<tr>
<td>Concrete</td>
<td>33 73</td>
<td>23 56</td>
</tr>
<tr>
<td>Diagram</td>
<td>5 11</td>
<td>1 2</td>
</tr>
<tr>
<td>Mixed Mode</td>
<td>4 9</td>
<td>4 10</td>
</tr>
<tr>
<td>Verbal</td>
<td>3 7</td>
<td>13 32</td>
</tr>
<tr>
<td></td>
<td>n=45 100%</td>
<td>n=41 100%</td>
</tr>
</tbody>
</table>
Observations during data collection indicated that Task 1, the Step Problem, provided the greatest challenge for students. Hence the responses for Task 1 were examined more closely in relation to quality. Based on the SOLO Taxonomy, statistically significant differences were recorded by grade for SOLO response to Task 1 (p<.05). As the data in Table 2 show the number of verbal responses increased for the Grade 10 group but the figures for the number of relational and better SOLO responses are lower with the Grade ten group.

Table 2: Summary data for Task Strategy and SOLO Level by Grade for Task 1

<table>
<thead>
<tr>
<th></th>
<th>GRADE 7</th>
<th>GRADE 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unistructural</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>Multistructural</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>Relational</td>
<td>16</td>
<td>21</td>
</tr>
<tr>
<td>Extend Abstract</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>Missing</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>n</td>
<td>45</td>
<td>41</td>
</tr>
</tbody>
</table>

School View

Consistent with the analysis for the whole grade view results were examined most closely in relation to Task 1, the Step Problem. Although the cells were too small for statistical comparisons summary data for the two schools in Table 3 indicate that School A recorded substantially more verbal responses to Task 1. However, when combined with mixed format responses the figures are similar for both schools. But SOLO levels for the responses indicate that the School A cohort was more inclined to produce unistructural responses than the School B cohort. The result raises questions about motivation as well as cognitive ability.

Table 3: Summary data for Task 1 Strategy and SOLO Level by School

The next step was to look more closely at the school results by grade and gender in relation to Task 1. Table 4 provides an indication that the School A group recorded more verbal responses and that boys were
marginally more likely to respond verbally than girls. However, the differences do not appear to support any substantial gender differences.

When broken down further by grade and school (see Table 5: 5a and 5b), the data show some shift away from concrete responses but perhaps not so much as one might have expected after the additional three years of instruction.

Table 4: Responses to Task 1 by School and Gender

<table>
<thead>
<tr>
<th>SCHOOL A</th>
<th>GIRLS</th>
<th>BOYS</th>
<th>SCHOOL B</th>
<th>GIRLS</th>
<th>BOYS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete</td>
<td>10</td>
<td>9</td>
<td>13</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Mixed</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Verbal</td>
<td>8</td>
<td>10</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Totals</td>
<td>22</td>
<td>24</td>
<td>21</td>
<td>19</td>
<td>46</td>
</tr>
</tbody>
</table>

Table 5: Responses to Task 1 by Grade, School and Gender

Table 5a: Grade 7

<table>
<thead>
<tr>
<th>SCHOOL A</th>
<th>GIRLS</th>
<th>BOYS</th>
<th>SCHOOL B</th>
<th>GIRLS</th>
<th>BOYS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete</td>
<td>10</td>
<td>8</td>
<td>7</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Mixed</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Verbal</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Totals</td>
<td>12</td>
<td>13</td>
<td>10</td>
<td>10</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 5b: Grade 10

<table>
<thead>
<tr>
<th>SCHOOL A</th>
<th>FEMALES</th>
<th>BOYS</th>
<th>SCHOOL B</th>
<th>FEMALES</th>
<th>BOYS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Mixed</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Verbal</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Totals</td>
<td>10</td>
<td>11</td>
<td>11</td>
<td>9</td>
<td>21</td>
</tr>
</tbody>
</table>
Discussion

Responses to the small scale space tasks reinforce the findings from the pilot study that students, regardless of year group, have a strong preference for processing problems in concrete formats. In relation to Tasks 1 and 2 the Grade 7 group recorded fifty-three percent and seventy-three percent respectively of their responses in concrete format. In the grade 10 group the concrete responses were thirty-nine percent and fifty-six percent respectively for Tasks 1 and 2. While there are some differences between grades especially for Task 2 the trend remains and raises questions about emphases in mathematics teaching strategies within the schools. As students have reported in feedback this preference comes from the additional information that appears to be available when you can 'see the shape' rather than 'hold it in memory' either in the presentation format or in the problem solving process.

What is surprising is how few students appeared to show signs of cognitive development in their visual problem solving skills with patterns and relationships. While it seems reasonable to assume that Grade 7 students might respond in lower SOLO levels it is somewhat mystifying why Grade 10 students should have relatively fewer SOLO responses in the extended abstract levels. Twenty-three percent of the Grade 7 responses were classified as extended abstract compared with ten percent of the Grade 10 responses.

When the data were analysed by school the pattern became somewhat clearer. In School A, twenty-three percent of responses were classified as unistructural compared with five percent for School B. This seems to be suggesting a difference between the school groups and the possibility of neighbourhood differences as a potential source of explanation. Given the interests of this research in spatial diversity of groups of people and the influence of place on conceptual development the finding is important for pointing directions for further research.

Large Scale - Photographic Tasks

All responses related to the questions designed for the photographic tasks were examined closely for the location marked on a photocopied version of each representation. Most responses had the Thompsons located in a feasible position relative to the clustered settlement. However, explanations varied and six categories were identified based on the criteria shown in Table 6. Phenomenographic techniques (Marton, 1981) were used for this process.

Comments illustrative of each explanation category were as follows:
Note: For consistency examples are taken from Year 10 students at School A

- Rent concept only
  "houses not built as well"; "Doesn't come with many facilities"; "quality of the house - brick or weatherboard"; It would depend on where the area is. If it was a swamp or good neighbourhood"

- Mathematical / arithmetic
  "Each time went down by $200"

- Contextual - simple relationship
  Its cheaper further away from the city

- Contextual - plus explanation
  "these [urban locations] are like suburbs the others are like rural and therefore lower"

- Contextual - plus understanding
  "Rates for things like water, land. Dearer for people to live where there are lots pf houses and where there are not so many houses it gets cheaper. Closer to the city centre it is more expensive" "Living in the centre of town - nearly on the main road. No 2 a bit out of town. No shops, neighbours, right out of town. No 2 is an OK spot. Can get their supplies quickly - might have a shop - a small one - corner store. Step 3 on edge of town. So might not have a shop - might have to drive all the way. Have a few neighbours. Maybe a few supplies in the front of the house. Stop 3 out whoop whoop might have to have supplies driven out to them"

Table 6. Category Frequencies of Students' Responses to tasks

In relation to the request to illustrate their understanding with a graph or sketch the responses were varied and somewhat disappointing especially for the Grade 10 age group. As indicated in Table 7 half of the Grade 7 students and more than one third of the Grade 10 students were unable to conceive the relationship in graphical form. In retrospect this outcome has been looked at carefully in relation to the interview technique. Questions to be considered are how much does this poor response reflect the formal curriculum and how could the question be posed for a more effective set of responses.

Table 7. Category Frequencies of Graphical Interpretations of Meanings
Discussion

As shown in Table 6 considerable variation occurred in responses to the photographic tasks by the two school groups. Most notable for School A is the rather large number of responses related to a rent concept of house prices rather than a land-rate concept. This and the number of nil responses compared with School B which had no nil responses raises questions regarding prior knowledge of students in these two locations.

The Land Rent Principle as an essentially linear relationship is one that is strongly embedded in a philosophy that relates to land and building ownership. Therefore, it should have been of no surprise that children who have lived most of their lives in public housing will relate more easily to the concept of 'rent' rather than 'rates'. The Land Rent Principle in geographic terms, or the notion of rates as used in the problems posed, is likely to have academic meaning only to a student who has lived his or her life in a rent paying public housing estate. The concept 'rent' in such circumstances is linked to such variables as 'how nice the house is', or 'does it have a view?'.

Students' graphical representations of students' understandings of the photographic information relative to rates is interesting because of the variety in responses that the qualitative approach allows. Again, phenomenographic techniques were used to determine categories of responses. As shown in Table 7, responses recorded for all groups (N=86) ranged from simple one variable, two dimensional graphs to two dimensional three variable graphs, and sketches. However, an overwhelming number of responses, thirty-six percent, were single dimensional with little apparent improvement in the complexity of responses due to maturation between school years 7 and 10. For the Grade 7 samples, eighteen percent produced graphical responses with two dimensions and for fifteen year olds the response level was twenty percent. In terms of the SOLO taxonomy this means that students were operating primarily at the functioning modes associated with concrete and ikonic reasoning rather than at the level of abstract reasoning where typically concepts can be manipulated and new hypotheses explored. Maybe the fifteen year olds sampled were slightly younger than the age group typically associated with the appearance of these higher order thinking skills but given that many of the students at both schools spoke of continuing their education to tertiary or university levels the absence of such skill is surprising. One encouraging aspect for this research are the creative interpretations of graphical representation. There is evidence of predominantly mathematical symbolism in a number of the versions and in others there is more attention to pictorial symbolism.
SUMMARY

In the research described an investigation was made into the ways in which adolescents process visual stimuli in relation to tasks designed in the context of mathematical or small sale space and geographic or real, large scale space. And with previous research the findings indicate the strong preference for students to process problems in concrete formats and with very little progression with age. Groups of students selected from school located in two contrasting neighbourhoods and from Grade 7 and Grade 10 (N=86) showed considerable similarity in their patterns of responses especially in relation to the small scale tasks. Some evidence of school based differences existed in terms of SOLO levels of responses and in the percentage of verbal format responses. These did not appear to be gender or age related.

Responses to photographic tasks showed more clearly the influence of background knowledge and lived experience. An interpretation or land rates in terms for a rent concept seems entirely consistent for a group who live predominantly in rented public housing.

To make conclusions regarding students' preferences for teaching strategies that use concrete, three dimensional solid forms when teaching patterns and relationships seems to have some merit. However, this research is in its infancy and we need to explore these issues in much more detail and greater depth and in many more social and geographic contexts. Already, though, we see the value of reconstructing children's meanings using their action and words as a path to better informed teaching and learning practices.

REFERENCES


