Researching Mathematical Understanding Through Children's Representations of Number

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This paper provides an overview of recent studies in mathematical education exploring the relationship between children's representations of number and their conceptual development. Our observations are interpreted with respect to developing theoretical models for mathematical learning and problem solving based on characteristics of representations. Children's external representations are categorised according to three dimensions: (i) the type of imagistic representation identified by pictorial, ikonic and notational recordings; (ii) the level of creative structural development, and (iii) evidence of a static or dynamic nature of the image. In some cases the representations revealed idiosyncratic, highly individualistic images. We draw on our data collected from a number of studies in early number learning: a cross-sectional study of 166 children in grades K-6; a 2-year longitudinal study of children's multiplication and division strategies; and related teaching experiments.

In this paper we address the question of how children's representations of mathematical ideas are linked to conceptual development. It has not yet been established exactly how children use their representations in building mathematical concepts. We attempt to throw light on this question by drawing on our empirical data from recent studies on counting and numeration, and multiplication and division processes.

Representations and Conceptual Development

A broad conceptual basis for number concepts and arithmetic operations develops fundamentally from the child's experiences over a period of time with a range of problem-solving situations, and from establishing relationships between these situations (Davis, 1992; Hiebert & Carpenter, 1992; Vergnaud, 1988). We now have an extensive body of research indicating that even young children are able to construct number concepts and solve arithmetic problems by using representations in physical, pictorial, ikonic or notational forms. Recent studies suggest that children's representations of problem-solving situations are closely linked to their conceptual understanding and the way children construct mathematical relationships (Hiebert & Wearne, 1992; Lampert, 1992; Maher, Martino & Alston, 1993). The child uses existing mental representations, and may be required to extend these or construct new representations in order to gain a solution (Davis, 1984). This may promote the development of new mathematical ideas. Children's representations also reveal much about the idiosyncratic and creative ways in which they structure mathematical relationships (Maher, Davis & Alston, 1991; Thomas, Mulligan & Goldin, 1994).

A number of researchers suggest that conceptual understanding is built on the notion of constructing connections between representations of mathematical ideas (Hiebert & Wearne, 1992; Hiebert & Carpenter, 1992; Lesh Post & Behr, 1987). This involves building relationships between quantities and action on quantities that are represented physically, pictorially, verbally and symbolically (Hiebert & Wearne, 1992). Moreover, they suggest that building connections between external representations supports more coherent and useful internal representations (Hiebert & Carpenter, 1992).

Representational Systems

In this paper we draw upon a number of theoretical models for describing internal and external representational systems. Piaget and Inhelder (1967) inferred some kind of internal representation of spatial concepts when they referred to representation or imagination involving the evocation of objects in their absence. These

representations were considered to be constructed through progressive organisation of motor and internal actions, resulting in the production of operational systems. The early work of Piaget & Inhelder considered the importance of images as internal, holistic representations of actions which can be inspected and transformed.

The SOLO Taxonomy (Biggs & Collis, 1982) has been widely used to analyse the structure of solutions to a broad range of mathematical problems presented by students. The SOLO model builds upon Piaget's stages of development, and this includes an ikonic mode of response related to children's internal perceptual pictures. The use of imagery is central to this process of representation.

Kosslyn (1983) defined four types of image processing: generating an image; inspecting an image to answer questions about it; transforming and operating on an image; and maintaining an image in the service of other mental operations. These processes are all concerned with the construction of the child's internal representation systems. One problem is to determine how accurately these internal representations can be inferred from external representations that the child might produce under varying conditions.

Goldin (1992) distinguished cognitive representational systems internal to problem solvers (a theoretical construct to describe the child's inner cognitive processing) from (external) task variables and task structures. We draw the distinction between external representations (a structured environment with which the child is interacting that may include, for example, actual physical objects to manipulate and actions in response to that environment), and internal imagistic representations (a theoretical construct to describe the child's inner cognitive processing).

In presenting our data, we consider three of the five types of internal representational systems discussed by Goldin: (a) verbal/syntactic systems (using mathematical vocabulary, developing precision of language, self-reflective descriptions); (b) imagistic systems (non-verbal, non-notational representations, e.g. visual or kinaesthetic); and (c) formal notational systems (using notation, relating notation to conceptual understanding, creating new notations).

These systems develop over time through three stages of construction: (i) inventive/semiotic, in which characters in a new system are first given meaning in relation to previously-constructed representations; (ii) structural development, where the new system is "driven" in its development by a previously existing system; and (iii) autonomous, where the new system of representation can function independently of its precursor.

Dorfler (1991) is also concerned with the role of imagery in doing mathematics. He suggested that an important component of mathematics education is for the learner to be encouraged to make external his or her internal images. He suggests that meaning is induced by concrete "mental images" and that protocoling one's own actions in specific situations and processes generates a representation of the concept.

Pirie and Kieren (1992) have proposed a new model for the growth of mathematical understanding that reflects a notion of personal building and reorganisation of one's own knowledge structures. The model involves eight potential levels or modes of understanding for a specific person with respect to a particular topic. It attempts to link imagery with the development of understanding. Any level of understanding has embedded in it all other more inner levels of understanding. Growth of understanding is described as a dynamic organising process whereby extending knowledge involves both abstracting one's understanding to a new outer level and folding back to recursively reconstruct one's inner level knowledge. From a state of 'primitive knowing', levels of understanding progress through 'image making', 'image having', 'property noticing', 'formalising', 'observing', 'structuring' and 'inventising'. At the level of 'image having' the child is able to manipulate and use the image in mathematical thinking. 'Formalising' entails consciously thinking about the properties of images, whereas 'structuring' means being aware of one's formal observations in terms of a logical structure. This relates similarly to Goldin's notion of imagistic systems and the phase of structural development.

Pirie and Keiren's model brings together a number of contemporary views of how children think about and represent their mathematical ideas. There is more interest in the modes in which children think and represent concepts rather than stages they progress through. The child's use of imagery is seen as central to the theoretical frameworks described thus far. We now draw upon some studies researching the role of imagery and number.

The Role of Imagery in Representation of Number

The role of imagery in the representation of mathematical ideas has been described by a number of researchers (Bishop, 1989; Clements, 1982). For example, Hershkowitz and Markovitz (1992) emphasised the importance of visualisation of mathematical concepts and the development of advanced visual thinking. Hershkowitz (1993) and Bobis (1993) investigated the role of visualisation in estimation of number. Hershkowitz showed that visual imagery played a vital role for 9 year olds in doing numerical tasks, especially in problem solving. Bobis (1993) found that with practice, Kindergarten children were able to use visualising strategies to mentally combine and separate patterns. The children developed subitising skills and started to relate number patterns mentally as they enhanced part-part and part-whole relations. Ten frame imagery was found to be a useful referent for children in their visualisation of number.

Recent work (Brown and Presmeg, 1993; Brown and Wheatley, 1990; Presmeg, 1986, 1992) in which individual students' thinking was probed in clinical interviews, indicated that students use imagery in the construction of mathematical meaning. Brown and Presmeg (1993) assert that learning frequently involves the use of imagery although sometimes it might be very abstract and vague forms of imagery. Presmeg (1986) identified five types of visual imagery used by students: (i) concrete, pictorial imagery (pictures in the mind); (ii) pattern imagery (pure relationships depicted in a visual-spatial scheme); (iii) memory images to recall information; (iv) kinaesthetic imagery (involving muscular activity, e.g. fingers 'walking'; and (v) dynamic (moving) imagery involving the transformation of concrete visual images.

Recent findings (Brown and Presmeg, 1993) revealed wide differences in the types and facility of imagery used by students in problem solving. All students in the study of seven fifth grade and six eleventh grade students used some type of imagery to solve mathematical tasks. Students with a greater relational understanding of mathematics tended to use more abstract forms of imagery such as dynamic and pattern imagery while students with less relational understanding tended to rely on concrete, kinaesthetic and memory images.

Reynolds and Wheatley (1994) reported that fourth/fifth graders used recording to help symbolise mathematical constructions, and children's reflections on these symbolisations elaborated their mental schemes. They found that images were often not well developed but were constructed during reflection on an activity. They suggest that children benefit from these ways of externalising their own meaningful constructions, rather than having the symbolisations of others imposed on them.

Research on Representations of the Number System

Research groups focussing on children's counting strategies (Kamii, 1986; Steffe, Cobb and Richards, 1988; Steffe, 1991; Wright, 1991) and conceptual development of numeration (Bednarz and Janvier, 1988; Cobb and Wheatley, 1988; Denvir and Brown, 1986a,b; Fuson, 1990; Hiebert and Wearne, 1992; Kamii, 1989; Ross, 1990) have highlighted children's

construction of representations of the number system. Boulton-Lewis (1994) showed that children's levels of counting were significantly related to their knowledge of, and ability to explain, place value in the first three years of school.

Rubin and Russell (1992) assert that children's counting, grouping, estimating and notating skills are essential elements in developing representations of the number system. They describe these elements in terms of "landmarks in the number system". These landmarks appear to be related to additive structure, multiplicative structure, the generation and analysis of mathematical patterns and mathematical definitions. Rubin and Russell suggest that people who are adept with number operations e.g. computing, comparing, and estimating, have a non-uniform view of the whole number system.

Hiebert and Wearne (1992) investigated children's structural development of numeration. They describe children's understanding of numeration as "building connections between key ideas of place value such as quantifying sets of objects by grouping by ten, treating the groups as units... and using the structure of the written notation to capture the information about grouping" (p.99). Other studies have focused on structural aspects of numeration, identifying the child's ability to group and re-group composite units of ten and relate this to a general structure of the base ten system (Kamii, 1989; Cobb and Wheatley, 1988).

Children's representations of number as some form of physical, pictorial, or notational recording have been exemplified in many studies analysing children's structural development of number and understanding of the numeration system (Davis, Maher and Noddings, 1990; Goldin and Herscovics, 1991; Hiebert and Wearne, 1992; Hughes, 1986; Rubin and Russell, 1992; Thomas, 1992). Thomas (1992) reported the wide variety of mental pictures of the number sequence 1 to 100 that were used by 40 children in Grades K-4. Although some aspects of structure appeared in the imagery of Grade 2 children, most Grade 4 children still did not possess the structural flexibility with number to successfully mentally manipulate 2-digit numbers.

In a further cross-sectional study of 166 children (K-6) and 79 high ability children (Grades 3-6) it was found that the children's internal representations of numbers were highly imagistic, and that their imagistic configurations embodied structural development of the number system to widely varying extents, and often in unconventional ways (Thomas, Mulligan and Goldin, 1994). Children's active processing of internal images were found to be static or dynamic in nature: static meaning a fixed representation, and dynamic as a representation that is changing and moving. In the cross-sectional sample, 3% of the children displayed dynamic images of the number sequence whereas 10% of the high ability children used dynamic images.

An exploratory study of 77 high ability Grade 5 and 6 children investigated links between their understanding of the numeration system and their representations of the counting sequence 1-100 (Thomas & Mulligan, 1994). Analysis of children's explanations, and pictorial and notational recordings of the numbers 1-100 revealed three dimensions of external representation: (i) pictorial, ikonic, or notational characteristics, (ii) evidence of creative structural development of the number system, and, (iii) evidence for the static or dynamic nature of the internal representation. Our observations indicated that children used a wide variety of internal images of which 30% used dynamic internal representations. Children with a high level of understanding of the numeration system showed evidence of both structure and dynamic imagery in their representations.

Figures 1 to 6 show examples of children's representations of the number sequence. The children were asked to close their eyes and to imagine the numbers from one to one hundred. Then they were asked to draw pictures that they saw in their minds. They were also asked to explain the image and their drawing.

Figure 1 Anthony (Grade 1) Figure 2 Kimberley (Grade 2)

Anthony (Figure 1) drew a picture of a truck and explained the image as "cause my Dad's truck does a hundred". This external representation was pictorial and we infer an inventive semiotic internal representation relating the truck-image to speed. Kimberley (Figure 2) and Mellissa (Figure 3) produced ikonic representations of the number sequence. Kimberley's recording was of ten groups of ten circles but she could not identify the structure explaining her drawing as "just circles". Mellissa gave evidence of a highly structured imagistic internal representation for the developing numeration system with her drawing of ten, ten rods.

Figure 3 Mellissa (Grade 2)Figure 4 Robert (Grade 2)

Robert (Figure 4) drew a square and subdivided rows of separate squares, each square not being aligned to adjacent squares, and then recorded numerals for the numbers in squares, 1 to 17 being in the first row. This partial array displays an emerging notational structure, but Robert showed evidence of further difficulty with using ten as an iterable unit, saying "you just put the numbers in the boxes as far as you can go... and you count in ones". Jane (Figure 5) and David (Figure 6) both recorded the number sequence using conventional notation but in highly creative ways. Jane explained that she saw the numbers moving in a spiral formation "going on forever" and it is important to note the structure of segments of number strings in tens (e.g. 71-80, 81-90) that she used. David described numerals flashing one at a time, multiple counting in fives up to 100.

Figure 5 Jane (Grade 1)Figure 6 David (Grade 4)

It was inferred that each representation is closely linked to their conceptualisation of structure of the number system. Anthony has not developed structure or a notion of number sequence as yet, but has a sense of the importance and size of 100. Anthony, Jane and David showed highly creative imagery that was not related to their conventional experiences in the classroom. Kimberley, Melissa, and Robert produced images that reflected aspects of classroom experiences. We inferred from Robert's recording that his conceptualisation of the number sequence was linear although his external representation showed a partial array. He had some sense of the structure of an array being involved but this was not related with organising numbers in tens.

From analysis of a range of recordings we inferred that the more coherent and well-structured the child's external representations, the further developed was the structure of their internal representational system of number. We now draw on some related work on children's conceptual development of multiplication and division through representations of problem-solving situations.

Representations of multiplication and division

A number of recent studies on children's solutions to multiplication and division problems indicate that children as young as kindergarten age can solve a variety of problems by representing or modelling the action or relationship described in the problems. (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993; Kouba, 1989; Mulligan, 1992a,b). Kouba found that 30% of Grade 1 and 70% of Grade 2 children could solve simple equivalent set problems, while Mulligan found a steady increase in success rate on similar problems from over 50% at the start of Grade 2 to nearly 95% at end of Grade 3. Carpenter et al. (1993) found that Kindergarten children used modelling as an effective way of making sense out of problem-solving situations. However, it is not clear , from these studies how instruction can be based upon children's modelling and representations.

Other studies on teaching and learning multiplication and division have focussed on the various ways children represent multiplication and division situations (Burns, 1989; Lampert, 1990; Murray, Human and Olivier, 1992; Nesher, 1992; Rathmell and Huinker, 1989; Reuille Irons and Irons, 1989; Steffe, 1991). Lampert (1990) used an instructional approach based on problem solving intended to help children make sense of the process of multiplication and division with numbers larger than 10. This included telling and illustrating number stories, working on problems whose solutions require different ways of grouping and counting, and solving problems drawn from "real world" situations. An important feature of this work is how she encourages children to connect representations (that make sense to them), to the symbols used to represent these processes in formal school mathematics such as algorithms. Burns (1989) presented multiplication in contexts involving children in thinking about grouping, investigating patterns, writing multiplication stories, relating multiplication to geometry, and patterning on a multiplication chart. Another approach, focussing on part-whole relationships was implemented by Rathmell & Huinker (1989) investigating children's representations of multiplication and division problems through the use of materials and diagrams. Similarly, Reuille Irons and Irons (1989) developed multiplication and division strategies where children were involved in language experiences (talking, listening, reading, writing) through modelling, creating and sharing problems.

Mulligan (1992a,b) examined young children's development of multiplication and division processes through their representations of multiplications and division problem solving situations in a two year longitudinal study. This was followed by a problem-centred teaching project with third graders (Mulligan, 1991), and a follow up study of sixth graders (Mulligan, 1993). A range of multiplicative word problems based on a classification used by Greer (1992), were presented by the researcher to children in a mini-classroom setting. Evidence of children's solutions to a variety of problems showed their verbal and written explanations, physical modelling of the solution, and pictorial, ikonic and notational recordings. There was a wide variety of representations used and children were able to solve multiplication and division problems without difficulty and with minimal intervention or direction from the researcher.

Structural development of multiplication and division

Children's internal conceptual structures of multiplication and division were inferred from elements such as grouping, partitioning, counting and patterning in their external representations, and their explanations of these representations. Representations of multiplication showed evidence of conceptual development at three levels: (i) equivalent grouping with unitary counting without the formation of a composite structure e.g. "four ones as four"; (ii) repeated addition of equivalent groups by transitional, double counting or adding; and (iii) multiplication represented as a binary operation. Strong similarities could be found in the recordings of representations of division. Levels of structural development resembled those found for multiplication: (i) a rudimentary level where children shared one by one and used unitary counting; (ii) building down where children worked down from the dividend successively taking away (e.g. 16, take 2, take 2....); (iii) building up where children used counting or adding to 'build up' to the dividend (e.g. 2,4,6,8,...16); and (iv) multiplication used as an inverse operation.

Figures 7 to 12 show selected examples of representations by Grade 3 children for multiplication and division problems. These examples attempt to show how the representations reveal developing conceptual structures for multiplication and division. Figure 7 shows Michelle's drawing of and equal grouping problem, "two tables with three children at each table". She formed equivalent groups but focussed on her image of the situation as "4 girls and 2 boys" using unitary counting and addition to calculate an answer. This was meaningful for Michelle, but her internal image and notation reflected only partial development of the structure of multiplication. In Figure 8 a repeated addition model of multiplication is more clearly distinguished. Natalie represented an equal grouping problem ikonically, grouping in a linear formation with repeated addition of fours, adding on four each time.

Figure 7 Michelle (Equal grouping) Figure 8 Natalie (Equal grouping)

Rebecca (Figure 9) used an inventive pictorial and notational representation of equivalent groups, revealing her visualisation of an algorithm to structure the representation in vertical form. Rebecca connected this to her previously organised representations for addition and subtraction. She generalised this form of notation to other representations for multiplication.

Figure 9 Rebecca (Equal grouping)

For the compare problem ("I have 3 pencils, Sue has 9 times as many") the equivalent grouping was represented differently showing the relationship, pictorially, between the two quantities. She uses multiple counting pattern to calculate the answer and linked this with her grouping and notation of multiplication. This signals the transition to a model of multiplication as an operation.

Figure 10 Susanne (Compare)

Array problems ("there are 3 lines, 7 children in each line") were represented in rectangular formations and calculated by simple unitary counting, skip counting or repeated addition.
Figure 11 shows how Catherine represented the problem ikonically using notation 7x3=21 using counting in threes rather than the operation of multiplication.

Figure 11 Catherine (Array)

There were strong similarities found within the representations for division problems. Michelle's representation is shown in Figure 12 of a subgrouping problem (6 apples shared equally between 24 people). Her concrete model and ikonic recording of "bundles of sticks" was rather unusual but upon careful examination, she represented the 4 equal cuts in the apples using sticks. However, it was obvious that she did not use halving and re-halving accurately as other children did. Her guess was correct but she could not symbolise the sub-division process using formal notation.

Figure 12 Michelle (Sub-grouping)

Samantha used ikonic and notational representations (Figure 13) of a partition problem (There are 12 people and 6 tables. How many children at each?) where the division symbol was correctly used showing a quotient of 2 even though formal division had not been introduced in the classroom. While Samantha solved this problem easily using mental representation, it was not until she recorded her solution that it became apparent that she was able to use division symbolism meaningfully.

Figure 13 Samantha (partition)

The quotition situation shown by Rebecca in Figure 14 enabled her to readily form equivalent groups pictorially and ikonically where she shows the structure of repeated addition to reach the dividend of 18. This "building up" process was prevalent amongst the solutions.

Figure 14 Rebecca (Quotition)

The external representations provided by children in these examples reflect the developing structure of multiplication and division.

Further analysis of each child's elaborations about their representations showed a more coherent picture of their conceptual understanding of multiplication and division across a range of situations.

Implications

Children's representations of the counting sequence, in conjunction with a range of numeration tasks revealed a wealth of information about their understanding of the counting sequence, their structure of numeration, the order and magnitude of numbers, and use of pattern and notation. Focussing on children's representations of multiplication and division provided valuable evidence of their counting and number fact knowledge, level of conceptual understanding, problem-solving strategies, intuitive models for multiplication and division, and the ways in which they made connections between their representations. Children did not need to master computational skills in order to solve these problems, nor did they require any particular training in problem-solving and representational strategies. They were able to work independently and co-operatively, explain and justify their representations, and express their problem-solving strategies in writing. Many representations in both the numeration and the number operation studies revealed a creative aspect of mathematics exemplified by invented symbolism and unconventional methods of solution. Other representations revealed misconceptions and immature strategies that might have gone undetected in a traditional mode of instruction.

There are also limitations in analysing representations. External representations form an incomplete picture of what is happening in the child's mind. We cannot ever be certain that the internal representations that we are inferring are completely accurate and so a variety of assessment methods over a period of time must be used to build-up a picture of the child's conceptual understanding. It is not sufficiently comprehensive to only look at specific tasks at one time and in one situation. A variety of methods are needed for children to produce imagery in the form of pictures, diagrams and notation and verbalisation of the meanings of these representations. Task based interviews and observations over a period of time in naturalistic settings may provide more explicit knowledge about children's changes in imagistic systems.

We raise the question of whether highly developed representational capabilities influence the way children apply their knowledge in mathematical problem-solving situations, and if so, what implications can be drawn for instruction? Of particular interest is that young children are able to use concrete materials to solve problems, while at the same time producing a variety of pictorial, ikonic and symbolic recordings.

Observations from our research studies on numeration and multiplication / division have been interpreted with respect to the theoretical models for mathematical learning and problem solving based on characteristics of representations. Examples of children's responses to problem situations have been used to infer connections between the external representations that the children produced and their growth of internal conceptualisation. We found that children naturally generate imagery as part of the construction of understanding, thus it is important that when developing structure for the counting sequence or the structures of number operations that they are encouraged to use their imagery.

The verbal syntactic, imagistic and formal notational systems that form part of the internal representational systems suggested by Goldin (1992) are seen to develop holistically and to be continually modified by experiences. Our data best reflected this model. However the growth of understanding that occurs in developing an autonomous conceptual state is well illustrated by Pirie and Kieren (1992). Their model involves the manipulation of existing imagery in mathematical thinking, reflecting on and making explicit externally one's imagery,

and developing awareness of structure in the internal representations. The idea of 'folding back' emphasises the non-linear nature of this growth and of the need for children to go back to inner modes of understanding when challenged with new mathematical tasks / questions that are related to familiar problem situations.

Further research, including longitudinal studies, needs to investigate how we can both assess and promote the development of conceptual understanding through encouraging children to represent mathematical ideas. If young children are able to solve arithmetic problems meaningfully by using different modes of representation, can curricula and assessment be designed to promote the development of mathematical concepts and problem-solving processes through representational thinking?

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