The Function Concept:
Making Connections within and between Representations

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The function concept is a central idea underlying secondary school algebra yet it is only recently that the mathematics education research community appears to be establishing an organised research agenda for discovering the difficulties that students face in attaining the concept. Three representations for function are currently dominant in our school mathematics curricula - the table of values representation, the algebraic representation and the graphical representation. The usefulness of being able to link these representational forms is unquestioned yet there is evidence that students compartmentalise each one and do not always draw on the most efficient form for problem solution. This paper considers recent research on student understanding of the function concept and points to the characterisation of student responses in the tabular representation in particular where patterning has been considered to be easily accessed by the novice student. It appears that much student patterning is pre-algebraic in nature and is strongly influenced by an arithmetic processing perspective which may actually hinder algebraic development for the student.

The function concept is a crucial one in secondary school mathematics (Dreyfus, 1990; Hamley, 1934; Kaput, 1992; Yerushalmy, 1991). It can be seen as a unifying idea throughout the algebra curriculum and research on student conceptions of function has recently become of international interest in the mathematics education research community (Wagner & Kieran, 1989).
Students have difficulties with various representations for function: set-theoretic, tabular, algebraic and geometric representations. However, it is the nature of the connections within and between representations that is worth investigation (Kaput, 1989). It appears that students maintain very separate representational systems for function and do not fully exploit the complementary features of each one. As teachers we appear to assume that connections are made simply; primarily when two or more representational systems are available to the learner. Deliberate teaching strategies that confront the difficulties in making the connections would be worth developing.

Mathematical Structures - Evolution

The internal structure of mathematics built up by each learner has been of particular interest to many mathematics educators. The possible match between conceptual development for humankind and the individual has been noted by Piaget among others:

ìPiaget ... draws a parallel between the historical and individual development of knowledge, and finds that evolution of individual thought sometimes follows the same progression as the history of scientific thoughtî (Ginsburg & Opper, 1979, p. 9).

Piaget was interested in a comparison between the evolution of key ideas in humankind and the evolution of the understanding of such ideas in the individual, relating to natural phenomena like ìthe movement of the clouds and of rivers, the problem of shadows, or the displacement of water when an object is immersedî (Ginsburg & Opper, 1979, p. 5).

One can trace elements of the relation/function concept in primitive humankindís match of time and the seasons which resulted in the development of agriculture, crucial to the advancement of the our species. ìThe notion of function, like that of relation, is one of the most elementary in human thoughtî (Hamley, 1934, p.5). Much algebraic teaching methodology relies initially on natural language as a translational medium to the formal mathematical form or syntax: ìas x increases by one, y increases by 2î is meant to lead to y=2x+c. However, strict dependency on natural language mediation has been identified as a root cause for misconceptions in algebraic understanding (Kieran, 1990).

The evolutionary perspective may bear fruitful suggestions for a consideration of the development of understanding of the function concept in studentsí mathematical growth. Kleiner (1989) speaks of the evolution of the concept of function dating back 4000 years with ì3700 of these consist(ing) of anticipationsî - certainly the anticipations being of paramount interest to educationalists as it is in the anticipations that we may find reflections of student development. Kleiner traces the recent 300 year development and continual refinement of the concept of function which he depicts as:

ìa tug of war between two elements, two mental images: the geometric (expressed in the form of a curve) and the algebraic (expressed as a formula - first finite and later allowing infinitely many terms, the so-called ëanalytical expressioní). ... Subsequently, a third element enters, namely, the ëlogicalí definition of function as a correspondence (with a mental image of an input-output machine)î (Kleiner, 1989, p. 282).

It is worth noting here that the geometric context came first. Leibnizís and Newtonís images were geometric. With time an ìincreased emphasis came to be placed on the formulas and equations relating the functions associated with a curveî eventually leading to them being ìindependent of the original curveî (Kleiner, 1989, p. 284). This appears to mirror some classroom development where the geometric image is either subordinate to the algebraic formula or, (at worst) non-existent. The complementary or supplementary roles of each image could perhaps be more positively exploited in teaching.

In relation to current accepted classroom practice, one is faced with many questions. Is one image treated as subordinate to the other? If so, what message does this convey to the learner with a visual preference for mathematical images? What is the implicit message with regard to the nature of ëlegitimateí or ësophisticatedí mathematics itself? Whose images are they anyway? The role of images in mathematical learning does, fortunately, take on new meaning with the advent of dynamic computer software and it is here that the teacher and the learner may, at the end of the twentieth century, have a chance to confront such questions.

Process or Object Conception?

The function as object and/or process is becoming a popular theme in the literature on function understanding (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Kieran, 1990; Sfard, 1992). Breidenbach et al. (1992) outline their general epistemology and relate it in particular to the development of an understanding of the function concept. Understanding is taken to go beyond manipulation of formulas to involve ëa process conceptioní that allows for the transformation of mental objects, that is the student has ìthe ability to construct processes in their minds and use them to think about functionsî (Breidenbach et al., 1992, p. 247). Both process and object conceptions could be significant mechanisms for understanding.

Rather than process conceptualisation being seen as the primitive, Schwartz and Yerushalmy (1992) argue that ì(t)he symbolic representation ... is relatively more effective in making salient the nature of the function as a processî while ì(t)he graphical representation ... is relatively more effective in making salient the nature of function as entityî (Schwartz and Yerushalmy, 1992, p. 263). Dubinsky and Harel (1992) alert their reader to the fact that the two conceptions are inextricably linked - ìit is important to point out that an object conception is constructed by encapsulating a processî (Dubinsky & Harel, 1992, p. 85). There are shades here of the van Hielesí (1986) model of thinking whereby a higher level of understanding is derived by perceiving the properties of a former level as a structural entity.

Context. Moschkovich, Schoenfeld and Arcavi (1993) believe that the problem context and interpretation play leading roles in determining which perspective is invoked. The process perspective involves the linking of x and y values and the object perspective involves the thought of function as an entity, either as a member of a parameterised class (in an algebraic representation), or as a graphical object which can be acted on wholly, so to speak. They argue ìthat developing competency ... means learning which perspectives and representations can be profitably employed in which contexts, and being able to select and move fluently among them to achieve oneís desired endsî (Moschkovich, Schoenfeld & Arcavi, 1993, p.72).

Preferred Representation

A natural or learned dependence on one or other representation may be a reason for student difficulties with the translation process between representations. It is a common experience for teachers to see their students floundering with the algebraic form when the graphical form in fact offers a more efficient problem solving strategy.

Current practice in Australian schools most usually develops the intuitive base for function with tabular representation with input-output value pairs (eg. ìGuess my ruleî). The tabular form may in fact be more complicated than we supposed. Regarded as an ideal early introductory form for showing the functional relationship between dependent and independent variables, the patterning we expect students to extract from the table could be either on an arithmetical plane or an algebraic plane. Patterning may be more complicated than we think: extracting algebraic form from numerical data involves a leap in perspective from individual paired data to global conception of all paired data, that is seeing the general in the particular.

Pre-algebraic and Algebraic Student Responses

A study by Kaput (1992) provides insight into significant empirical research on these issues. Kaput looked at the processes by which students construct and formalise patterns in numerical data (mainly with linear functions). His study is significant in that it involved an attempt to classify student responses. His work attempted to get at the ëfine-grainedí matter of understanding. He used the ëGuess my ruleí format where a sequence of functions was provided and his students had control over the determination of the sequential input values for each function. The output value was then given and the students had the option of guessing the function at any stage.

Rhetorical approach to algebra. From a group of secondary school students who had either one or two years of algebra, Kaput found students whose understanding he described as bounded by a rhetorical understanding of algebraî (Kaput, 1992, p. 290). This understanding was based on a ìnatural language based descriptionî which, for many students, appeared not to develop over time with standard school mathematical experiences.

A much smaller number of Kaputís students developed a ëpost-Vietaí sense of algebra where they sought the parameters that defined the function. Kaputís work details, at a micro-level, the impact of natural language-based processes in the externalisation of quantitative reasoning which was common amongst his less sophisticated students. His student sample exhibited two of the historical evolutionary stages of the use of symbolism.

Kaputís tasks. In Kaputís study students were presented with the ëGuess my ruleí computer program which had been pre-programmed with functions (four linear and two quadratic). The students chose a numerical input value for the function and were given the output value. The student was able to guess the function in the form Y= ... at any stage.

Characteristics of responses. Kaput identified some interesting characteristics of student responses:

ïArithmetical character: The arithmetical character of student responses was the most obvious feature - students used numerical approaches to find underlying patterns. Even with the graphical mode students disembedded the numerical information and worked from this.

ïPre-algebraic responses: These students focussed on very short sequences of data, often the last input match. They used a natural language description (eg. double the number and add one) and were unable to mentally vary the parameters of the function at hand. Functions, for them, were strictly process-rule and not conceptual entities. Kaput suggests they may have what has been termed an action conception - a more primitive conception than a process conception: they deal with only one single application of the pattern at a time and appear unable to grasp that the pattern must hold across multiple instances; algebraic symbolism is used as a shorthand descriptive language.

ïAlgebraic: A very small number of students were described as having algebraic type responses. They could look at a sequence of data as a whole and recognise the algebraic representation of that class of functions in terms of the parameters. This style appeared however to have been learned (a product of instruction) rather than spontaneous.

ïProcedural view: There was evidence also of a procedural view, again arithmetic in character - students responded with the rule in the form of Y = X(5) and the statement ìmultiply by fiveî. This was not surprising to Kaput given the nature of the studentsí previous mathematical experience where a primarily procedurally-based cognitive structure would have been developed.

ïNegative numbers: An interesting phenomenon was the difficulty pre-algebraic students had with negative numbers. Kaput conjectured that this ìdifficulty with negative numbers may turn out to be a reliable marker of pre-algebraic thinkingî (Kaput, 1992, p. 295).

ïGraphical representation: Almost all students began with the table of data mode. An accidental feature of the graphical mode uncovered pre-algebraic thinking. The graphical mode provided the students with all prior input-output pair points but at the bottom of the screen the last input-output pair was stated. Those students who were pre-disposed to using the last input-output match ignored the visual data in the graph (where, one would think, the straight line would have appeared to have been a strong influence) and, despite oral prompts, their strategy was mediated by natural language encodings of this last pair of data.

ïStability: Another finding of this study is, that despite the extended experience of the tasks where dozens of functions were used with the observer providing help at times of frustration, students seldom moved across a response class (pre-algebraic to algebraic). The stability of their response type was remarked upon and compared with the historical development of algebraic thinking which suggests that the difficulty of moving from arithmetic to algebraic thinking should not be underestimated.

Finding Patterns in Numerical Data

From Kaputís (1992) study it appeared that some students had considerable problems with the tabular representation of function and that further identification of the characteristics of student understanding is possible. The links that students make between algebraic form, graphs and tables of values is the focus of my own study. Kaputís results suggest that it may be possible to establish the difficulties that students, who find mathematics very difficult, encounter: those students whom Hart (1981) states ìcan deal with problems that require only one or two steps for a solution but any demand for abstraction or even the formulation of a strategy is beyond themî (Hart, 1981, p. 209). In this case, those students who do not develop a concept of function in any representation.

My own study seeks to detail levels of understanding of the function concept in students from Years 7 to 10 and to identify problems which hinder students from advancement in understanding. The test instruments include two pen-and-paper tests (Tests 1 & Test 2) which have been administered to 200 students from mixed ability settings. A later stage in testing will involve the tracking of student use of a computer program which purports to facilitate a multi-representational view of function.

Test 1 concentrated on understanding of the table of values representation, in particular knowledge of its structural conventions and derivation of patterning behaviour in a table. Test 2 concentrated on graphical understanding of linear functions in particular and the links made with the tabular form and the algebraic rule. Understanding of the Cartesian plane conventions and behaviour formed the core of this second test. Test 2 included some items which matched items on Test 1. Only the Test 1 instrument is discussed here.

Test 1 was concerned with (a) the tabular form and natural language description of patterns in the tables of values, (b) the derivation of an algebraic rule in the form y=Ö from tables, (c) recognition of co-variance in a table and, (d) recognition of global patterns in tables.

Section A of Test 1 looked at patterns in a sequence of numbers. Students were asked to complete various number sequences and identify increasing or decreasing patterns. These items were written primarily to uncover patterning perspectives, however, difficulties with negative numbers and fractions could also be identified from the responses here.

Section B of Test 1 consisted of written textual information where a contrived ëeverydayí context provided related numerical data. Students were asked to arrange the information in a table and then use that table for both interpolation and extrapolation prediction.

Section C of Test 1 asked the students to provide data in proforma tables according to natural language rules, eg. _ is three more than eq, and y is half x.

Section D provided tables of values and asked students to derive a natural language description of the relationship between the numbers in the x and y columns, a statement of the rule y=Ö, and corresponding y-values for three x-values that also belonged to the original table.

Sections E and F asked students to complete sentences of the form ìas x increases by Ö, y increases by Öî for data tables, and the reverse.

Section G asked students to match data tables which represented ìthe same relationship between the x and y valuesî.

From an initial assessment of student performance on Test 1 there were some interesting outcomes. The 200 scripts are yet to be fully analysed and matched with responses on Test 2, however there are some suggestions of similarities with student responses in Kaputís (1992) study. Many students appear to be operating pre-algebraically in the sense that they deal with only one instance of a pattern at a time (one input-output pair) and appear unable to grasp that ëthe patterní must hold across multiple instances. Dependency on numerical processing gave their responses an arithmetical character.

Initially, it appears that some students are unfamiliar or not confident with negative numbers, fractional forms (eg. y is a third of x) and squaring. This suggests that for some students their arithmetic confidence or experience needs to be enhanced before they are exposed to algebra. Kaputís students were also seen to be operating arithmetically or pre-algebraically.

The (learned) conventions of the structure of the tabular format were not well established by students. The usefulness of ordering the values of the independent variable x (eg. x=0,1,2,3Ö) to help to establish consequential patterning of the dependent variable y was not recognised by many students. In Section B, where information was to be arranged in a table, students rarely ordered the data. When students were asked to arrange data in a table and then use the table to predict other values, some resorted to the original problem context to sketch the new situations rather than derive information from their table.

Many of the responses could be seen to be pre-algebraic in the sense that arithmetic processing was not generalised for all pairings of (x,y) and responses tended to be isolated to one set of (x,y) pairings at a time. The global relationship appears not to be an early focus for students in their search for pattern.

It appears that some students did not grasp the fact that a table could establish the pattern of relationship between two stated variables. It was hoped that Test 1 would uncover studentsí understanding of the table of values format and in this sense it seems that some students viewed tables of information as generally unstructured forms. In particular, the need or usefulness of ordering one of the variables to see the pattern of the global relationship was not grasped by some students. This suggests that the structural features of tables may need to be emphasised more explicitly.

Natural language descriptions of the tablesí patterns also indicated that some students were operating pre-algebraically in Kaputís (1992) terms. Similarly, correct natural language description of a global pattern as ìadding threeî was unsuccessfully translated to the rule, ìy=+3î which indicates a procedural view of function in Kaputís terms. Functions, for these students, were strictly process-rule and not conceptual entities.

The subtleties of the tabular form are not easily seen. In grasping the underlying functional structure of a table of values, a student needs to see both correspondence and co-variance. This involve a double perspective of individual input-output match and overall matching of the two sets of data (x and y). I suspect that this may be very difficult for most students.

The data from Tests 1 and 2 allow for a fine-grained analysis and studentsí patterns of response on Test 1 are still to be compared with their response patterns on Test 2 which focussed on graphical understanding of linear functions and the links made with the tabular form and the algebraic rule. Profiles of student responses are to be assembled and will be analysed for patterns of behaviour across the three representations for function: tabular, algebraic and graphical. It is hoped that more specific information about learning in these representations will be derived from the next stage of this research where individual students, will be monitored as they work with a software package which purports to promote a multi-representational understanding of the function concept.

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Rhetorical algebra is the verbal or prose style of writing algebra, used before Diophantus (third century AD) who made a start on the use of algebraic symbolism (Baumgart, 1989). In the historical development of algebra as a symbol system this first stage of rhetorical algebra was followed in the sixteenth century by ìsyncopated algebra (where) the use of abbreviations forunknown quantitiesî (Kieran, 1990, p. 97). The third stage, symbolic algebra, was initiated in the sixteenth century by Vieta who was the first to use literal symbols for

coefficients in equations thereby enabling classes of equations to be solved. His symbolic algebra thereby enabled the writing of rules governing numerical relations.

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