Horses for courses: Ways in which children solve mathematical problems

ABSTRACT

This paper looks at the impact visual and nonvisual learning approaches have on the mathematical problem-solving performance of primary school children. It is argued that visual strategies are best utilised when solving novel problems, while analytical methods should be fostered as problem complexity leads to familiarity. In particular, this paper suggests that preference for solving mathematical problems either visually or nonvisually should not be attributed to "learning style", but other factors including problem type, complexity, novelty, solver understanding and success. The importance of creating a balanced curriculum where both visual and nonvisual learning approaches are given equal weighting is emphasised, along with the need to expose children to alternate solutions to mathematical problems.

RATIONALE

Preference for solving mathematical problems

Several mathematics educators have argued that some children have a preference for using "visual" methods for solving mathematical problems, while others prefer "verbal-logical" or "analytical" methods (Clements, 1984; Dreyfus & Eisenberg, 1986). Krutetskii (1976) argues that these two modes of thought provide different "mathematical casts of mind" that determine how an individual conceptualises mathematical ideas. In the same vain Suwarsono (1982) concluded that all people have a preference for either visual, nonvisual or a combination of both methods when solving mathematical problems. Other studies have found that students who prefer to solve mathematical problems either visually or nonvisually do so even when particular problems may be more easily solved using an alternate method (Krutetskii, 1976; Lean & Clements, 1981; Shackels & Eliot, 1981).

This raises the question whether problem solvers in fact have preferences in the methods they use to solve mathematical problems, or do other factors need to be considered? While some problems may be solved most quickly and efficiently using either visual or non-visual approaches (Denis, 1991; Presmeg, 1986), problem type, complexity, novelty, solver understanding, and success, need to be determined to understand fully the reason for a person choosing to solve a problem a particular way. It will be argued that children may solve difficult, complex problems visually, but solve problems that are easily interpreted or understood in an analytical way. Preference for solving problems may be related to the nature of the individual problem encountered, and not on an ability or learning style.

Suwarsono (1982, p.151) defined a visual method of solution as one that involves the use of any pictorial representation (diagrams or pictures), either on paper or in the mind. A nonvisual method being a solution which does not involve the use of pictorial representations. Suwarsono maintained that a nonvisual representation is conducted on the basis of processing mathematical statements using rules of mathematics and language. It is suggested in this paper that visual methods are best utilised when a student is unable to use or relate mathematical reasoning directly to a particular mathematical problem.

The importance of visualisation in solving novel mathematical problems

The importance of visualisation in education is now accepted. As Eisenberg and Dreyfus (1989) have maintained, few educators will dispute the role of visualisation in the curriculum, in fact, most will say that much will be gained from visually based approaches. It could be argued, nevertheless, that research should not focus on the classification of children as either visual or non-visual thinkers, or in the need to justify the importance of visual or spatial learning. Future studies must focus on the open-ended nature of visual and spatial problem tasks, and the flexibility that creates integration across the curriculum.

Mitchell & Burton (1984) emphasised the importance of using visual skills when solving difficult problems by stating that "although poor spatial ability can be compensated for, to some extent, by good verbal ability, one who is able to employ several strategies equally well has a better chance of success at problem-solving" (p. 396). Children who are able to use visual techniques to explore difficult problems may have more success in interpreting visual problems than children who are unable to draw diagrams, visualise a problem or "become part of the problem".

Kaufmann (1990) proposed a theory that describes the division of labour between linguistic-propositional and imagery representations along a novelty-of-task dimension. Kaufmann believed that a problem arises when processing does not run automatically. Great precision may be achieved in a linguistic-propositional descriptions of a task. Such a task is easily and quickly described and can demand a full range of computational operations. By contrast, tasks demanding imagery are likely to be more ambiguous, and it is difficult to direct children's thinking. This may, however, be necessary if lack of experience, or computational break down hinders performance. Imagery is particularly useful where the need for processing is high, as is generally the case with high task-novelty conditions. In such circumstances imagery can become a back-up system that provides access to a set of simpler cognitive processes of a perceptual kind. Limitations of conceptual operations may also result from the complexity of the information being processed.

Mayer (1992) suggests that many difficulties people have in solving problems can arise from using the wrong Schema (a schema being items in our long term memories that are connected to other related items). Often confusion, or bottlenecks, in our short term memory results.

Sweller (1993) found that novices often employ a means-end analysis to overcome such instances. Here the problem solver tries to extract differences between each problem state, to find problem solving connections that can be used to reduce or eliminate these difficulties. As Mayer & Sims (1994) maintain, students who possess the specific knowledge needed in that domain may not need visual aids to solve problems because thay can generate their own representations as they read or explore a task. The individual is generally selecting the approach which presents minimal cognitive load.

Other findings also provide grounds for the assumption that imagery demonstrates its efficiency in variable or novel tasks, but it may lose its impact when a problem can be solved more rapidly on the basis of propositional representations (see Denis, 1991; Kosslyn, 1983). The importance of visual imagery, however, should not be overlooked. The present study examines the role visual and nonvisual methods have on children's approaches to solving mathematical problems. More specifically, the current study investigated three categories of questions:

1. Do children prefer to solve novel problems in an visual manner?
2. Can children who solve a significant proportion of problems in an analytical, nonvisual manner also solve problems visually?
3. Do children who predominantly use visual methods to solve mathematical problems have difficulty proposing alternate (nonvisual) solutions to such problems?

METHOD

Procedure

Subjects were given eight mathematical problems from Suwarsono's (1982) Mathematical Processing Instrument. This instrument has been used successfully in studies that assess children's preference for solving mathematical problems (see Lean & Clements, 1981; Lowrie, 1994; Shackles & Eliot, 1981). The students were asked to solve the eight mathematical problems and then encouraged to provide an alternate solution or method to solve each problem. Children who were unable to propose an alternate solution were asked to draw a diagram to solve this problem or create a number sentence or pattern to express your findings?

Children who solved a problem visually where asked to solve the problem in a different manner (not using a strategy that involved the use of a picture or diagram), while those who solved the problem using an analytical method where encouraged to develop a visual method.

The questions

1. If the time is 8 O'clock in the morning, what was the time nine hours ago? (Make sure you include a.m or p.m as part of your answer).

2. Two families held a party. Three members of the first family and five members of the second family attended the party. Each of the members of the first family shook hands with each members of the second family. How many handshakes were there altogether?
3. A girl had eleven plums. She decided to swap the plums for some apples. Her friend who had the apples, said “For every three plums, I will give you an apple”. After the swap, how many apples and how many plums did the girl have?
4. A tourist travelled some of his journey by plane, and the rest by bus. The distance that he travelled by bus was half the distance he travelled by plane. Determine the length of his entire trip if the distance he travelled by plane was 150 km longer than the distance travelled by bus.
5. One day John and Peter visit the library together. After that, John visits the library every two days at noon. Peter visits the library every three days also at noon. If the library opens every day, how many days after the first visit will it be before they are, once again, in the library together?
6. Altogether there are 8 tables in a house. Some have four legs and the others have three legs. Altogether there are 27 legs. How many tables are there with four legs?
7. A saw mill saws long logs, each 16m long, into short logs, each 2m long. If each cut takes 2 minutes, how long will it take for the saw to produce eight short logs from one log?
8. Some sparrows are sitting in two trees, with each tree having the same number of sparrows. Two sparrows then fly from the first tree to the second tree. How many sparrows does the second tree have more than the first tree?

ANALYSIS OF PROBLEM-SOLVING METHODS

Students who used visual methods to complete the mathematical problems tended to draw detailed diagrams or pictures to solve the examples presented in this research. James, for example when solving Question 1, drew the following diagram.

John, on the other hand, who solved the problem by counting back 9 hours from 8:00 am in an analytical manner without drawing a diagram, drew the following diagram when asked to represent a visual response to the same question.

Although the solution is visually presented, his initial technique of counting back from 9 is represented in his diagram.

Other examples of the additional detail represented in an initial visual solution, as apposed to a visual solution being developed after an analytical solution are indicated in the methods used by Daniel and Kristie in solving Question 2.

Daniel's initial visual response   Kristie's visual response after initial analytical response

Almost all children (94%) who solved the problem analytically were able to support their answer with an alternate visual solution. Most children were able to do this quickly, with diagrams and drawings that were generally less detailed and accurate, than first attempt visual responses (as mentioned above), but often with similar diagrams to those proposed by first time visual thinkers. Melissa, for example, solved Question 3 quickly using the following method.

11 ÷ 3 = 3 apples r 2 plums.

She, however, was also able to demonstrate an alternate visual method.

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           apples                plums
 1_____2____     3           |
000      000     000      00
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When solving Question 4 Stephen also demonstrated an ability to present an analytical solution then a visual response when solving the same problem. Analytically he divided the journey into three equal sections, two sections being travelled by the plan, and one section by the bus. He then wrote 3 ¥ 150 km = 450 km. Asked to solve the problem another way he drew the following diagram.

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0  50  100  150 200 250 300 350 400 450 500 550 600
|___|___|___|___|___|___|___|___|___|___|___|___|
         Bus      |
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Stephen solved two problems by initially using a visual method. In

Question 5 Stephen drew a diagram to represent the days of the week, and letters to symbolise the days the boys visited the library.

J J
P___J___P___J______P
1 2 3 4 5 6

He was also to express his solution in an analytical manner.

2 ¥ 3 = 6
3 ¥ 2 = 6

When asked to explain this solution, Stephen said that John visits the library every two days, and Peter visits the library every three days, they would meet again on the sixth day.
To solve Question 6 Stephen again used a visual method first. He drew 27 lines to represent the table legs and grouped these into lots of 3 or 4 until it worked out evenly.

|||| |||| ||| ||| |||| ||| ||| |||
4 4 3 3 4 3 3 3

He couldn't immediately solve this problem another way, but eventually produced the following equation, which he explained in great detail.

3 5
¥ ¥
4 3
= =
12 + 15 = 27

You find two numbers that have several factors that equal 27 (12 + 15 = 27). Hopefully one of theses numbers has 4 as a factor, and the other has 3 as a factor. In this case 3 ¥ 4 = 12 and 5 ¥ 3 = 15, so you have 3 tables with 4 legs and 5 tables with 3 legs.

While this is an analytical response, it appears that Stephen's reasoning is based on "working backwards" from an answer obtained from a visual method. Stephen was asked to create a problem that would allow him to use such reasoning (this method) to solve a similar problem but could not. The analytical response given was achieved through the manipulation of numbers that "held true" for this particular problem. Stephen needed to draw a diagram to solve this problem. An alternate method was achieved only after an answer had been obtained.
While Stephen was able to construct both visual and nonvisual solutions to all the mathematical problems many children, however, could not

propose alternate solutions to problems after first using a visual method. In fact, of the thirty children interviewed, 64 (27%) of the 240 questions were only solved visually.

Glenn highlighted the importance of being able to use visual methods when confronted with a novel problem. He solved the eight problems in a nonvisual manner but incorrectly answered two questions. When asked to draw a picture or diagram to help interpret the problem, was able to rectify his mistakes. When trying to solve Question 7 he assumed that the log needed to be cut 8 times to produce 8 pieces, resulting in an equation 8 ¥ 2 minutes = 16 minutes. Glenn was asked to draw the log, and found that only 7 cuts were required to produce 8 sections.

He changed his answer to 14 minutes. When asked what strategies he usually used to solve mathematical problems he responded " I would draw pictures if I couldn't understand a problem in the usual way. I wouldn't draw pictures if I understood the problem". He hadn't drawn pictures to solve the two problems answered incorrectly because "I thought they were easy and didn't need to use them". Such responses highlight the positive effect visual methods play in the interpretation of mathematical problems.

IMPLICATIONS

It seems the methods children use to solve mathematical problems are not based on "innate" preference or learning style. In the above cases the students generally used analytical methods if they interpreted the conceptual processes of the problem quickly. The longer it took for a student to interpret the information of the problem, the more likely the child would use a visual method to complete the problem. Competent mathematicians usually used visual methods after studying the problem for several minutes.

Less competent children didn't usually spend more than 2-3 minutes studying a problem before providing a solution, usually in a nonvisual form. These children, when confronted with a difficult problem, would often use numerals presented in a problem, and create a number sentence using an operation (+, -, ¥, ÷) to complete the problem, almost hoping they chose the correct sign. It is important to assist students to become "part of the problem" so that problem-solving processes are meaningfully administered. Encouraging students to visualise problem situations may lead to a better understanding of what the problem is actually asking. Teachers should encourage visual thinking, particularly when it is evident student understanding of the problem is low. As Dreyfus (1991) states:

in order to give our students the chance to profit from and to appreciate the power of visual reasoning we, as a profession, need to upgrade the status of spatial reasoning in mathematics; in our own mathematical thinking, we need to generate visual arguments, to learn how to examine their validity and to accord them parallel weight as we accord to verbal and formal arguments (p. 43).

Children must be exposed to a more flexible learning environment that promotes the use of visual thinking. Students should be equipped with both analytical and visual problem solving techniques that attempt to foster a more balanced approach to interpreting, analysing and solving mathematical problems. Classroom teachers have the responsibility of encouraging children to “think more visually”, and need to be dedicated to such a pursuit.

CONCLUSION

Three points have been developed in this paper that challenge the way problem-solving approaches are fostered in primary school mathematics curricula.

1. Teachers must “believe” that visualisation has an important role to play in the problem solving process. Children must be encouraged to draw diagrams, create pictures in their mind, and generally experiment with a variety of solutions.
2. Competent problem solvers were inclined to solve complex problems in a visual manner, particularly when faced with novel examples. Interestingly, weak problem solvers tended to not use visual methods when solving difficult problems. When asked to draw a diagram or picture, success rates increased.
3. Students who solved problems in an analytical manner were generally able to provide a alternate visual solution to the same problem. Visual methods were not transferred to nonvisual methods as readily. This may suggest that visual methods are important when task-difficulty is high, with nonvisual methods best utilised when complexity leads to familiarity.

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