Mathematica
In Australian Education

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Introduction. I will take a two pronged approach. My overriding interest is to discuss the opportunity computer environments for mathematics provide to make a mathematical description of a person's world more accessible and relevant. In order to enter this discussion at a concrete level, I will explore this opportunity in the context of a particular environment - Mathematica.

Animating Mathematics. 100 years ago life was thought of by some as consisting of a body and soul. Today others consider self-replicating computer algorithms as representative of life. Mathematics, the Queen of the Sciences, by most mathematicians today is thought of as the logical derivative of some axiomatic set theory. With the increasing integration of computers into mathematics will set theory still reign, or will set theory prove to be but an embryo from which mathematics will mutate and continue to evolve? Will new computer environments provide the context for the next mutation of mathematics?

Mathematics Computer Environments. Set Theory traditionally has been taught as the foundation or most basic concept of mathematics through the modern period. It's emphasis is on defining membership - What are the solutions of a simultaneous system of equations in some algebra? - looks for the members of the algebra that satisfy each of the constraint equations. Category Theory a minor candidate for foundations took a different approach, it focused on the relationships between mathematical objects - the existence of a common solution to a system of equations could equivalently be identified as the existence of some relationships between the various objects or equations.
Mathematics computer environments have their own intrinsic somewhat different characteristics. Lists are the computer analog of a set - more particularly they are a finite ordered set, but the symbols they list need not represent finite constructions. Any mathematics computer environment provides a wide range of tools for the manipulation of lists. The basic purpose of a mathematics computer environment might be described as: to provide an environment for the analysis and representation of data lists. What type of analysis? By design the computer environments are developed to help us do what we already can conceive of doing. Not always things we have done, but at least things that we can conceive of doing. Interestingly new interpretations or observations often emerge from an actual, as compared to a merely conceived interaction with analyses and representations of data. These observations often influence future design. Today, after a couple of generations of interaction with mathematics computer environments, the environments have developed into a sophisticated, appropriate and general environment for representing and analysing lists of data. Within such environments the traditional analytic

methods, involving detailed calculation, have all but lost their relevance for the vast majority of students wanting to explore their particular profession's mathematical models.

With the introduction of the calculator many educators lament the lack of analytic calculation skills in their students. Will students of the future know how to calculate a derivative, integral, solution of a system of equations or even graph functions? Clearly mathematics education is at a critical point. A point at which some significant change in curriculum is imminent.

At such a time it is imperative that we again ask the question, why do we do the things we do the way we do? In general mathematics has become a tool for the analysis and representation of data. This' is, to many mathematicians, a boring part of the mathematical landscape. To others it provides the vital connecting point with the human situation, it makes mathematics relevant.

From the point of view of those who wish to use mathematical descriptions as a tool in helping them analyse and represent the world around them, calculation is not the primary skill they require. In the past it has been an indispensable skill, necessary to bring any mathematical description to its conclusion. Today, detailed calculation skills are rarely a necessary tool for someone wanting to apply mathematics to their world.

What do those who want to exploit mathematical descriptions need to know? They must know something about the situation they wish to model. The more formal or mathematical their structuring of their situation, the easier it will be to abstract from the intuitive a mathematical description. So skills at mathematically ordering their world are useful. These are not skills we spend any large amount of time developing at the moment. In addition, the more they know about possible off the shelf mathematical models, the more likely they are to find that an appropriate model exists for their problem. What would be helpful is an environment where a whole supermarket of custom models could be brought off the shelf. If no off the shelf model is available then tools for building an appropriate model would be useful. So, libraries of tools would be a great help to anyone trying to develop a custom model. Once the model is built, in goes the data, out comes the result. But is it correct? Skills in evaluating the validity of a particular model and calculation would be crucial. And then, the model is often only an approximate representation, other models can be developed, and the relationships between the various models and the real world situation explored, in a co-operative interdisciplinary environment.

How do these skills relate to current mathematics education? The answer is simple, there is very little relationship. Granted the mathematics computer environments have only just become affordable (Choice - Spend $250 on a child's toy that teaches them to calculate 2+3, etc,... or the same amount on a friendly computer program that does this plus provides a total mathematics computer environment for all their possible modelling requirements, right through to academic research?). But, they have been coming on line for some time, and there has been very little response from mathematics departments burgeoned with traditionally trained analytical

mathematicians that on the average have no computer training and are largely unaware of how to refocus their presentations away from calculation to modelling!
Like most sectors of society, mathematicians too have their difficulties, both personal and structural, in implementing change.

Mathematics and Object Environments for Education. The new human-machine environment for mathematics education can be naturally embedded into a much broader human-machine education environment. The schema in the figure below has been developed around the theme of education as moving from interaction with one's world, to structuring, to representing, to presenting one's experience to others.
In the new object environments, objects can be seamlessly mixed. One can relate to any number of different objects at once, or move in and out of the various object's contexts - never losing touch with the relationships one has introduced between the objects one has defined.

Appropriate Mathematics. There are at least two distinct approaches that can be taken when incorporating a mathematics object into mathematics education. Firstly, the mathematics object can simply be an appendage to the traditional curriculum - a new specialist tool for elucidating traditional material where appropriate. Secondly, the mathematics object may be allowed to reinterpret what is appropriate mathematics - the mathematics object defines the curriculum. In both situations human thought has shaped the universe of discourse. The first approach emphasises the historical analytical formalisation of our world. The second, channels enquires within the domain of a computer implementation. The approaches need not be exclusive, but it is interesting to consider the differences in each perspective.
What approach is most appropriate from the point of view of the student? The traditional approach has often alienated students from mathematics. Mathematics is not the primary field of study for the vast majority. Students often do just enough to give them the utility they need in their primary areas of interest. What do these students see as appropriate mathematics? One suggestion is that appropriate mathematics is relevant mathematics. Students often want to see the application of what they are learning. Traditionally, it has been difficult to present real world problems because of the complexity of real world calculations. Idealised problems have been explored. The actual application has been left to the specialist - who has some years of training ahead before they can actually approach the problem at any realistic level. When only concepts, not calculations are the barrier to developing a model of a real world problem, arguably the threshold to modelling a real world problem has been lowered. For example, at the tertiary level I have spent some years teaching first and second year students linear algebra. We say that linear algebra can be used to model many real world problems. But, what do our students know of

these. Much time is spent teaching them how to calculate various constructs within the theory. The abstract examples are chosen so as to illustrate properties of the theory. Even though they are never going to be required to calculate any of the constructs we introduce, they are told that such calculations will improve their understanding of the theory. While this is true - it is analogous to telling someone they need to become a mechanic to drive a car. It might be helpful, but it certainly is not necessary. Moreover, dragging students through the exercise takes valuable time away from looking at other possibly more relevant questions - like teaching them how to apply the theory to real world problems, allowing some mechanical engine to actually compute the result. Mathematics education training can move from less relevant calculation to highly relevant modelling of appropriate real world problems.

Learning to Model. Modelling moves the mathematics education setting away from the traditional mathematics environment towards application environments.

What is modelling? Modelling is the representation of a situation by some model. Mathematical models are formal representations of real world problems. Why model? One model can represent many different real world problems. If the one model is well understood, then one has a way of analysing many different often complex situations. Because the model is formal, the specific relationships described by the model can be calculated by a computer. So, if one can translate a real world problem into a mathematical model which has a friendly computer description, then one may have a powerful and efficient description for investigating real world relationships.

What does modelling in a mathematics computer environment involve? One needs to know how to translate the real world problem into a formal computer model. To do this one needs to know something about the formal model, something about the real world problem, and something about the process of translation.

A model is a particular interpretation of some theory. Underlying every formal model is a formal theory. There are at least three levels associated with a formal theory: The often imprecise intuitive interpretations of an individual which result in the formalisation of collections of concepts into a particular theory, an exploration of the interpretations of the theory and their relationship to the motivating model, and the application of models of the formal theory to particular contexts which includes an examination of the relationships between application models as different interpretations of the formal theory.

Intuition. A model of sufficient merit usually motivates the definition of a particular formal theory. This principal model while not a unique interpretation of the theory is usually the most natural interpretation for those defining the theory. However such an interpretation may not have even been thought of by those seeking to learn a formal theory. The question - What do you mean by that? - can often be helpfully answered by introducing the student to the concepts behind the definition - the principal model, which originally motivated the definition. This introduction can be best achieved when the interpreter relates to the student in their own natural language(translates the theory for their setting).

To reinforce the model other applications can be explored. The model may be closely related to other models - these models and their applications can also be explored. If the investigation of the model takes place without an interaction with the real world problems that motivated it, the model is dull and lifeless. Yet with the preoccupation with calculation in traditional mathematics curriculum the process of modelling is rarely fully explored.

So a first step, in learning to model might be a natural language discussion centred on examples and applications.

Model. Proficiency in using a particular model then results from becoming acquainted with it. Once the student can assign some meaning to the language of the model, they can begin to become familiar with the relationships this language describes. Feedback is still needed, but at a different level. The student can be directed to consider/or choose to interact with certain relationships described by the model. This process is somewhat mechanical, having learnt the stroke the student must swim some laps to increase their proficiency. A interactive multi-media platform provides appropriate technology for this mechanical task. After some initial work and with some maintenance, a co-operative environment can see the students themselves becoming the on-going developers and educators of each other in such a system. The hypermedia component provides an unhindered opportunity for students to contribute to their own education as well as providing for communications, skills development and testing (as well as student defined applications assessment) - enabling a more efficient use of human resources.

This could be described as the second stage in learning to model.

Applications. With the emergence of mathematical models unanticipated applications often emerge. Maturity in a formal model can be measured by the ability of a student to apply the model to an application. Indeed if the model is to have any real use to the student, then they must be able to discern areas where it might be an appropriate tool. A guided small group setting where each student is actively involved in corporately applying models to real world problems would be appropriate. To be able to apply a model and interpret the computations for a particular application is the culminating sign of a students conceptualisation of and proficiency with a model.

Success in this exercise would be a highly rated third and final stage in the process of learning to model.

These three components: an introduction to the intuition and the setting, skills acquisition and development, and application implementation form three complimentary elements in a students education in learning to mathematically model real world problems.

Learning to Model - Simulation. Mathematica implementation - computer simulation.

Model Land. Mathematica implementation - computer simulation.

Mathematics as Relationships. With the emphasis off calculation, an approach that emphasises relationships more than particular objects can be explored. For example: Linear relationships are represented by numbers (fields), matrices and vectors(modules over fields) and plane geometry. Traditionally the emphasis has been on detailed computations involving objects - evaluate a matrix determinant, compute a matrix inverse, etc - rather than appreciating or indeed discovering the properties of an object and its relationship to other objects - for example, the solution of equations relationship to matrix determinants, to the independence of vectors, to linear transformations, to the intersection of planes(space spanned by two independent vectors). This approach is possible because the mechanical task of calculating matrix determinants, solving linear equations, evaluating independence using cross and dot products(for a different approach), visualising transformations, and graphing planes can all be quickly and easily done within a mathematics computer environment. Such an approach encourages the student to look for interrelationships between, on the surface, very different objects. The stress is not on learning to compute, but on concepts and relationships. An analogous shift is observable in computer science - from machine language to compilers to high level object oriented programming languages - the programmer now focuses not so much on the detail, but more on elements of design. Modelling then is not just about exploiting some model for a particular application - modelling is a creative synthetic process of re-expressing real world relationships mathematically.

Exploring Relationships. Relationships in implementations - computer simulation.

Structural Disjunctions or Vertical Harmony. Let's say we spend the $250 on a mathematics computer environment for our three year old daughter. Mum shows her how to turn the computer on. On the desktop appears a number of objects or icons. Double-clicking any of these will take her into a new world. There are a couple of mathematics objects on the screen. Wisely she chooses to double click one of these objects. The object she chooses is in fact a document created by a mathematics computer environment. The object opens, what is she presented with? She sees an interface that will become familiar, for all the documents created in this one mathematics computer environment look similar. Graphics confront her. She tries to double click these - they come alive. They're animations - mathematical representations of sound, and pictures, and yes numbers! Her journey into a mathematical computer environment has begun. Each time she enters, she may choose to go via a different route or document; she may go deeper - maybe from wondering - what are those funny words and symbols that always seem to be associated with those terrific animations saying?
She is in a very complex environment - but the natural interface disguises its complexity. The same environment used not just by researchers applying

mathematics, but by mathematicians exploring the unchartered mathematical landscape. If she can use this environment as her tool for exploring mathematical relationships over the next 15 odd years of formal education before she is qualified to be a researcher, she should be quite comfortable with a very powerful environment. In reality she will likely explore way beyond where her predecessors travelled, possibly along quite unexpected paths, by the time she reaches this point.
If an educator wanted to use the computational power of the environment, but present the problem in another way, then they could use any number of tools to put together a front end for a mathematics exploration which had embedded into it the more general mathematics computer environment. Interested students need not be limited by this structure, they could have the option of moving into the underlying more general mathematics computer environment.
Now, suppose a particular school embraced these concepts, and alone restructured their curriculum. If at some point their students have to move into a traditional educational context, then they may be totally unprepared for the traditional calculation based work requirements. Conversely, should students in the calculation environment, move into a modelling environment they too will flounder. This was precisely the case when in 1990 at The University of Melbourne, the top students were introduced to the traditional theory, but using a problem based approach. During this trial I took a tutorial of students who had 100's and high nineties in both Year 12 maths, they struggled in this new setting, and yearned for a more familiar approach.
Conclusion, certain structural elements will need to be addressed to optimally implement any mathematics computer environment.
Vertical Harmony. The introduction of computer environments for mathematics would appear to direct mathematical enquires in new directions and at new levels. The conceptual framework of set theory with its emphasis on calculating membership, although necessary may not be as appropriate as the more general higher level approach of exploring relationships, and allowing a computer environment to evaluate the particular elements associated with any given relationship.
A vertical integration of one computer mathematics environment would maximise the opportunity of the large majority of students to utilise both their individual and the environment's awesome potential.
We are yet to see the results of a generation of mathematicians skilled in exploiting computer environments. What questions would they ask? What formal theories would be developed? What models would emerge? To what aspects of the human situation will formal representations be applied? How will they shape the way we make sense of the world about us?

Current Trial Programs. I am coordinating a number of Co-operative Research Projects in the primary, secondary, and tertiary settings. It's early days yet, but the enthusiasm of students and teachers is enormous. For those who are interested, you're welcome to become involved.
In each case the aim of the different projects is to explore in some

context the implementation of mathematics computer environments, and evaluate it against traditional and other possible options.

Programs

Computer Mathematics Environment Survey - October 1992.
Describing Your World Mathematically series.
Mathematica In Australia Journal.
Mathematica In Australia Conference: 6 - 9 December, 1992.
Chisholm College Braybrook - Year 11 Space and Number.
ACCEL, Chiaro Primary School - CME's for Primary Mathematics.
Victoria University of Technology - Mathematical Foundations (1st Year).
Mathematica as a professional mathematics object - current professions include: Electrical Engineering, Civil Engineering, Business, Botany.

Author. Dr Stephen M Hunt, was trained in pure and applied mathematics at Monash University, Clayton, Australia. At Monash he completed his doctoral thesis on the development of the theory of Modules over Inverse Semi-groups. Currently, a Lecturer in Mathematics at Victoria University of Technology, his research interests are in the development of computer environments, philosophy and algebra.
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