what might a fraction mean to a child and how would a teacher know?

gary davis, La Trobe University
robert P. hunting La Trobe University
Catherine pearn, La Trobe University
e-mail: matged@lure.latrobe.edu.au
fax: (+ 61 3) 478 7162

A paper presented at the joint conference of the Australian and new zealand associations for Research in Education
Deakin university, GEELONG
NOVEMBER 22-26, 1992

what might a fraction mean to a child and how would a teacher know?

introduction

The teaching and learning of fractions is not only very hard, it is, in the broader scheme of things, a dismal failure. Hart’s (1984) article documents the understanding of fractions by children in the 11 - 16 age range: the results are depressingly negative. What could be at the bottom of this difficulty to learn fractions, or to teach them well? The research literature on fraction learning is extensive. However it is painfully clear that this literature falls into small subgroups, with authors from one subgroup rarely citing the results of others. The situation is reminiscent of that in a branch of biological science described by Crick (1988, p. 158): “It tended to fall into a number of somewhat separate schools, each of which was rather reluctant to quote the work of the others. This is usually characteristic of a subject that is not producing any definite conclusions.” This is not to attribute blame to the dedicated and capable people who work in this area: it is our belief that it simply indicates no real progress is being made.

Categories

The process of categorization is basic to the activity of counting: without categories of things there is nothing to count. Recent theories of Changeux (1985) and Edelman (1987) have been proposed to account for various aspects of brain development and organization, including the phenomenon of categorization. For our purposes we need to know that categorization is a process that begins early in our lives, initially without the mediation of words, and is pervasive in most of our everyday activities. In the development of children’s counting types proposed by Steffe, von Glasersfeld, Richards and Cobb (1983) and by Steffe, Cobb and von Glaserfsfeld (1988) categorization is a basic mental activity that is absolutely essential to the act of counting. In our discussion of fraction meanings for children a basic rôle will be played by units of one sort or another. We hypothesize that these units, to be recognized as such by a child, must be seen by a child as exemplars of naturally occuring, usually socially mediated, categories.

Unit structures and structured units

By a unit structure we mean a group of one or more things that, taken together, form a whole - or unit - in which the parts stand in a particular relation to each other, for a specific collection of people. Examples from everyday life are: a football team, a social club, a building, an office, a country. Each of these things, for one reason or another - reasons that make human sense, to some people at least - are often regarded as single wholes, even though they are patently composed of constituent parts. As entities consisting of constituent parts they are, in particular instances at particular times, structured by the relation the parts bear to each other. These relations are social givens - attributes of a consensual domain (von Glasersfeld, 1991, p.xvi ; Cobb, Wood, and Yackel, 1991). In our terms, in order for a unit structure to exist there must be an agreed on relation, between the constituent parts of the whole, which is mediated between at least two, and usually many more, people.
A structured unit is a unit object - an instance of a category - that has an imposed structure of a particular sort. This imposed structure is one that comes from the mind of a person and which exists in the mind of that person. The person as actor may indicate this mental structure by signs or marks, and then by actions might convert the structured unit into another sort of object which we call a proto-unit structure. For example, if a person were to indicate f(2,3) of 6 counters they might conceive of each counter as a structured unit, as indicated below:

Then they might physically act so as to turn these unit counters with a mentally imposed structure - that is, these structured units - into objects that might appear to the reader as unit structures:

If they do so appear to the reader it is because the parts of each former unit counter stand in a certain relation to each other and, to the original unit counter, in the mind of the reader. In this physical action of cutting, say, the pieces of each former unit lie in relation to each other as parts of that unit. This is what makes the previous structured unit into a proto-unit structure. Without the relation between the parts the 3 pieces of the former structured units would not, in our terms, consist of the necessary ingredients for a unit structure. Notice now, however, that the relation that the 3 parts shown bear to the original unit, shown below, is a relation in the mind of the actor. It is not necessarily a socially negotiated relation, and it may be a relation that is difficult or virtually impossible for an observer to detect. So a proto-unit structure is like a unit structure, except that some relation between the constituent parts of the whole exists, so far as we can tell, solely in the mind of a single person: at least it is not clear that such a relation exists in any other way.

It follows therefore that an observer watching such a marking and a cutting might not be aware of the mental relation between the 3 cut parts that makes a proto-unit structure for the actor. However they may well feel that there is some relation in the mind of the actor which they, the observer, have yet to fathom. This, we believe, is the situation in which many child-observers find themselves when they watch a teacher-actor performing fractioning acts.

What converts a hypothesized proto-unit structure into a unit structure, for us, is a mediated communication process in which two or more people build expressed relations between the parts of a whole.The unit structure is then a unit structure for those people, but not necessarily for anyone else. In other words, experimental epistemological considerations enter in a concrete and basic way in the didactics of unit structures: as a teacher one must ask “How - that is, by what means - do I know this is a unit structure or proto-unit structure for that person?” This is an empirical, not a philosophical question - it does not have an a priori answer, and needs to be tested experimentally. To make an assumption about children’s internal mental relations is to beg the question didactically.

Fractions as operators

A framework

Fractions can be viewed as operators on natural numbers (Lambek, 1966) and a number of authors have used or discussed fractions of operators in didactic settings (Kieren and Nelson, 1978; Kieren and Southwell, 1979; Freudenthal, 1983; Davis, 1991; Hunting, Davis and Bigelow, 1991). We can view fractions this way as linear operators on submodules of the ring of integers. However it is equivalent, if somewhat less formal, to describe (positive) fractions as [m for n] operators, where m and n are counting numbers. For example a [1 for 2] operator acts on certain collections of discrete items - those that contain a multiple of 2 items - and produces a new collection of discrete items that has 1 item for every 2 there are in the original collection. Similarly a [2 for 3] operator acts on collections of discrete items that contain a multiple of 3 items to produce a new collection of items in which there are 2 items for every 3 there are in the original collection. Notice that a [1 for 2] operator acting on a collection of 4 items does not “take away” 2 of the 4 items: rather it “duplicates” or “clones” 2 new items. This way of thinking about proper fractions is entirely analogous to an operator view of whole numbers. Thus, a [2 for 1] operator acts on any collection of discrete items to produce a new collection in which there are 2 items for every 1 in the original collection. Notice that the information about what size collection an [m for n] operator can act on is given to us by “n”: the collection must be an integer multiple of n. In this sense whole numbers and general fractions can be viewed from a single perspective.

This way of viewing fractions as operators on certain collections of discrete items allows us, as adults and teachers, to think about certain structural aspects of fractions in ways that are appear to be a little different to the usual interpretations (as ordered pairs f(m,n)). For example, we say that two fractions are equivalent if they produce the same result on one (and so any) collection on which they can both act. This is the usual notion of equivalence of fractions to produce rational numbers: an [a for b] operator is equivalent with a [c for d] operator when they produce the same size collection of discrete objects by acting on a discrete collection of size b*d (for example; more generally, on a collection that is an integer multiple of the least common multiple of b and d). It is important to understand that in talking about fractions as operators in this way we are not pre-supposing an interpretation in terms of ratio and proportion. It is our belief that we are describing operators that are logically and psychologically anterior to ratio operators. As we will indicate below, the collection of fractions as operators is structurally isomorphic with the positive part of the set of ordinary fractions with usual fraction addition , multiplication, and order. It is only when we regard operators as the same when they are equivalent , as defined below, that we can be said to have constructed the rational numbers.

We can carry out two operations on these operators.We call the two operations “together with” and “followed by”. We will illustrate these operations via an example and leave it to the reader to formulate them in general terms. We consider the operators [1 for 2] and [1 for 3]. In order to define the operator “[1 for 2] together with [1 for 3]” we first have to have a collection of items on which the original two operators can act. This must be a multiple of 6. Then the “[1 for 2] together with [1 for 3]” operator acts on this collection of items to produce a new collection of items: first we have 1 item for every 2 in the original collection and then we have another 1 for every 3 in the original collection.

The operator [1 for 2] together with [1 for 3]] is then an [m for every n] operator: in fact, it is a [5 for 6] operator. The operator [1 for 2] followed by [1 for 3] is obtained again by acting on a discrete collection of 6, or a multiple of 6, items. First the [1 for 2] operator acts on the collection of 6 items to produce a new collection; this is then followed by the [1 for 3] operator that acts on this newly produced collection to produce a further new collection.

Again this is an [m for n] operator, namely a [1 for 6] operator. We could, of course, more conventionally, write:

- “[1 for 2] together with [1 for 3]” as “[1 for 2] + [1 for 3]”, and
- “[1 for 2] followed by [1 for 3]” as “[1 for 2] * [1 for 3]”.

Then our “calculations” tell us that [1 for 2] + [1 for 3] = [5 for 6], and [1 for 2] * [1 for 3] = [1 for 6].

Just as importantly, we can define an order relation between [m for n] operators. So we say that the [a for b] operator is smaller than the[c for d] operator when for a collection on which both can act, the first produces a smaller collection than the second. Thus, a [1 for 3] operator acting on a collection of 6 items produces a collection of 2 items, whilst a [1 for 2] operator acting on a collection of 6 items produces a collection of 3 items, Notice that in checking whether the [1 for 3] operator is smaller than the [1 for 2] operator this way we actually tell by how much smaller it is: namely, it is smaller by a [1 for 6] operator.

Notice that this way of comparing two fractions as operators is obtained by comparing two whole numbers, namely the whole number outputs when these fractions as operators act on a common discrete collection of items. This seems to us to focus attention on a much more important and salient aspect of a fraction (as operator or otherwise) than on the “numerator” or “denominator”. After all, if the numerator and denominator are the only salient features of a fraction then why should children not focus on whichever of these seems pertinent to a comparison problem? The fractions as operator idea hones in on the salient whole numbers to compare when comparing fractions. So to ask: “Which is bigger, a [3 for 5] operator or a [5 for 8 operator]?” we let both operators act on a collection of discrete items and compare outputs. Such a collection will have to contain a multiple of 40 items. For 40 items, the [3 for 5] operator will clone 24 items, whilst the [5 for 8] operator will clone 25 items. So the [3 for 5] operator is smaller (by a [1 for 40] operator). In order to decide such issues this way we see that we need to be able to know some whole number facts: the result of multiplying 5 by 8, how many times 5 and 8 divide into that, and what 3 and 5 times those numbers are respectively.

Empirical results on iterable units: moving froM f(1,n) to f(2,n)

Children as actors

It is all very well for us to talk about [m for n] operators as if the operating were just happening magically, without human intervention. We hypothesise that children can develop an understanding of fractions as operators in the sense we have outlined above, provided that they act on appropriate environments and can internalise those actions as mental records. In order that children appropriate mental records of what we regard as fractions as operators we believe it is necessary that they themselves do the operating. In this sense we see the notation [m for n] arising as a device for recording meaningful communication between teacher and child, and between children. The didactic setting we imagine for fractions as operators has a number of linked components to it. One of these components is a plenitude of “m for each n” situations and problems. Examples from everyday life are legion:

- Dad is nailing shingles on the roof. He will need four nails for every shingle.
- Mom is making a pie. She will use one kilogram of meat for every four people.
- Joe is making lemonade. He will need one lemon for every two glasses.
- Jenny is making clay dolls. She will need two button-eyes for every doll.
- Shelley is building a barbeque. She will need one fire-starter for every two logs of wood.
- Brian is a bus driver. He will have to collect one dollar from every person on the bus.
- We are going to feed the ducks on the lake. We will take one piece of bread for every two ducks.
- With these power boards you can attach five appliances to every power outlet.
- For this week’s supermarket special you save 5 cents on every cake of soap.

Another element of the didactic setting that we see as important is an adequate action tool. Such a tool should enable students and teachers to act and talk about fractions in a way that allows children to build

personal meaning. In the empirical results outlined below we have used two computer environments. One is the commercially available draw and paint program SuperPaint for Macintosh computers.This software provides a rich platform for action and reflection in a fraction and whole number domain. The other computer environment is a Hypercard instantiation of fractions as operators, through a virtual machine that we call the CopyCat.
We detail below two teaching episodes with grade 3 children at Bulleen Primary School, Melbourne, in which we use SuperPaint and the CopyCat.

Alisha and Sukey: a transition to f(1,3) as a number.
The following excerpts are taken from a teaching episode with two girls, Alisha and Sukey, on September 18, 1992. The excerpts indicate that in the context of the computer environments Sukey was comfortable with the unit fractions f(1,3) and f(1,5). However Sukey seemed less sure about f(2,3) which Alisha talked about in an insightful way. Both girls had difficulties with f(2,5). These excerpts, together with earlier teaching sessions with Alisha, suggest to us that whilst the girls interpret unit fractions of discrete collections of items by a division process, they have difficulty interpreting fractions in which the numerator is not 1. In particular, Sukey seems not to have made a connection that f(2,3) is twice f(1,3), whilst Alisha seems to come to this insight during the teaching episode. In previous teaching sessions Alisha had calculated f(2,3) of a number of discrete items by first dividing the collection into thirds and then subtracting from the total number of items. In other words, she had calculated f(2,3) n as n - f(n,3).
At the request of the teacher, Alisha made 11 copies of a small filled-in square in SuperPaint. This gave a total of 12 squares in all, but on the screen it appeared there were only 2 squares: the original and a copy. This is because 11 squares were stacked in a pile. The teacher asked Sukey to check there were 12 squares altogether. She put out 4 in a row, then 4 under that in a slightly staggered row, and then 4 spread out under that. This was a significant sequence of actions: it indicated to us that in her mind Sukey had already organised the 12 squares as 3 lots of 4. Why should she have done this? There was nothing in the question to indicate that she should organise the 12 squares in any particular way. Sukey appeared to be actively organising the data in front of her in her mind, and possibly anticipating a scenario to follow. This is a very important point: her actions - carried out in silence - indicate an active organising mental stance.The teacher asked the girls to make f(1,3) of the collection of 12 squares in a different pattern.
Teacher: What I'd like you to do is to make some more in a different pattern.
The girls made 1 new square of a different pattern.
Teacher: Now what I want you to do is to make as many of those as would make one third of all those. I want you to make a third of the 12 just that same colour there.
Teacher: How many do you have to make?
Alisha: 3 I think?

Alisha: Yeh, 2.
Sukey went to use the copy and paste function
Teacher: Paste, yeh, paste. Now see how many you've got. You've got 2. How many more do you want?
Alisha: 3. No, 2 more
Sukey: 1 more
Alisha: 1 more.
Sukey made another copy.
Teacher: Now, is that number there a third of those there, is it?
Alisha counts the 12
Sukey: No.
Alisha: No.
Sukey made 1 more copy.
Teacher: You think that's a third now?
Alisha: Yes.
Sukey: Yeh.
The girls had then made a total of 4 new squares of a different pattern. The excerpt indicates to us that both girls were hesitant about interpreting f(1,3) in the context of the 12 squares on the computer screen but came to a conclusion, by themselves, that 4 squares gave f(1,3) of 12 squares. Could they give a convincing reason for their final choice?
Teacher: How could you convince me it's f(1,3) ?
Sukey: Because there's three 4's in 12.
The teacher then copied the 12 squares to produce 24 squares. The girls were asked how many squares there were now, and both answered 24. The problem for them now was to find f(1,3) of the total of 24 squares. Their answers indicate that they both came to a rapid correct conclusion, but by different routes.
Teacher: Suppose you had to have f(1,3) of the whole lot? Now how many would you have to have?
Sukey: 8!
Alisha: Yeh, 8!
Teacher: Why 8?
Sukey: Cause eight 3's are 24.
Alisha: It's double - last time it was 4 and it's just double that.
Alisha's answer indicates that she sees f(1,3) of 24 as twice f(1,3) of 12, because 24 is twice 12. In other words, she seems to be thinking proportionally.
The teacher then removed 12 squares so that 12 squares remained on the screen. The problem posed to the girls now was to find f(2,3) of the 12 squares.
Teacher: Suppose I wanted f(2,3) of those 12. How many would I have?
Sukey: 8.
Alisha: Yeh, 8, 8,8,8,8.
Alisha: When you halved it, when you put it into f(1,3)'s before it was 4 so another one that's 8.
Teacher: Another one? Okay.
Alisha's answer is very significant and revealing: it indicates that she is

capable of iterating the f(1,3) operation in order to calculate f(2,3) of 12. Her excitement in stating “8” loudly five times in succession suggests hat she had a sudden insight. We had not observed Alisha to make this connection in previous teaching sessions. This observation of Alisha’s is quite different to her apparent proportional reasoning in finding f(1,3) of 24 as twice f(1,3) of 12. What she seems to have done is to realise f(2,3) of 12 as twice f(1,3) of 12. This is significant because it indicates that she can double f(1,3) and therefore may be thinking of it as a number.
Sukey’s answer was more difficult to interpret:
Sukey: You’ll put it all into 3 and then you take take away one, and plus the two other left -there’s 6, ah 8.
Teacher: Put it into 3 and take away one. That’s 4 is it?
Sukey: No! Yeh! And then you plus another one that is left, and it is eight.
She seems to say that f(2,3) of 12 is 12 -take away one-third of 12, as Alisha had in previous teaching episodes. However, she does not perform the calculation 12-12xf(1,3) but instead recognises that there will be two lots of 4 left. The teacher is somewhat confused and asks about the “one” being taken away. Sukey says no it is not 4, but then appears to understand the teacher’s concern and says yes.
The teacher placed 15 small squares of one pattern and 1 square of another pattern on the screen. The problem posed to the girls was to make f(1,5) of the 15 squares in the same pattern as the single square.
Teacher: How many would I have to make if I’m making this colour but I’m making f(1,5) of all those?
Sukey: 3.
Teacher; Why 3?
Sukey: Because three 5’s are 15.
The girls then copied the 15 squares. They were asked how many squares there now were, and both answered 30. The problem set to them was to say how many copied squares there would be if we copied f(1,5) of the 30 squares.Both girls used the same type of strategy they had used for finding f(1,3) of 24 squares:
Teacher: Well how many would there be now if I wanted to make a fifth of all them?
Sukey: 6
Alisha: Yeh, 6.
Teacher; Okay, do you want to tell me why 6?

Alisha: Because double what we did before.
Sukey: Because six 5’s are 30.
When they were asked how many squares would make f(2,5) of 15 Alisha said 12 and then 10. When asked how they could check, Alisha pointed at the Copy Cat and said “That.” When the teacher asked how she replied: “I don’t know. Make a f(2,5) machine.”
Elliot: a transition to relational thinking
The general problem posed to the children at Bulleen Primary School on Friday September 11 was this: if you had 24 pieces of chocolate (as

illustrated by a SuperPaint graphic) and were able to place this number of pieces on the in-tray of a CopyCat machine set to F(2,3), would the machine go, and if so, how many pieces would come out?

We will consider the case of Elliot and analyse his responses to problems about F(2,3) - specifically the afore-mentioned problem, a follow-up problem the same day involving 30 pieces of chocolate, and a similar problem posed the next teaching session which occurred on Tuesday September 15.

In the first episode Elliot and Shannon were together. Shannon is very bright, quick, and dominating. Elliot had a good grasp of whole number relationships for Year 3, but was not as adroit as Shannon, often preferring quick guesses without apparent reflection. The two boys were seated together at the computer with the teacher nearby. Because of Shannon's habit of rushing out an answer the teacher requested that he remain quiet while Elliot worked at the problem.

Despite encouraging comments made by the teacher and observer, Elliot did not respond to the problem of whether a F(2,3) machine would operate on 24 pieces of chocolate. The problem was restated. Elliot still offered no comment so a hint was provided by the teacher. The hint took the form of a question about how the CopyCat processes items when set to the fraction F(2,3). Elliot knew that the machine made little noises, and he offered 8 as his answer. Shannon correctly said 16, and then checked his answer on the machine. The teacher gave Shannon another task to work and worked individually with Elliot.

The new problem for Elliot was “Will the machine (set to F(2,3)) work if 30 balls are placed on the in-tray?” Elliot thought about this momentarily then answered “yes”. Further, he stated that 22 balls would come out.

Elliot held to the idea that 22 was the answer, despite various attempts by the teacher to get him to demonstrate the basis for his answer. He was first invited to justify his answer using the SuperPaint array which consisted of 30 square shapes. He was then given 30 MAB cubes. Elliot began to arrange them into groups of three, then groups of two, counting by twos. He said “twenty-two and four left” (that is, four groups of two).

The teacher attempted to focus Elliot back to the actions of the machine - “do what the machine does”. He even structured the cubes into 10 columns of three for Elliot. Despite repeated encouragement for Elliot to “do what you think the machine would do”, Elliot again insisted on 22. This time he counted out the first seven columns and partitioned the eighth column to give 22 cubes.

The teacher responded at this point by telling Elliot that 22 was not correct. He also substituted the question F(1,3) of 30 items. For about a minute Elliot did not respond. After Elliot failed to respond when asked what F(1,3) meant, a hint was offered whereby 6 items were input to the CopyCat and the Go button clicked upon. The teacher tried to focus Elliot’s attention on how the machine dealt with the input items rather than the output.

Teacher: What happened?

Elliot: Um, six went in and two came out.

Teacher: Yeh, but how did the six go in?

Elliot: It went in...

Teacher: You remember how it went in? Let's do it again.
Observer: Listen to the noises Elliot.
Teacher: Pay attention
Six items were input.
Elliot: When um, when three went away one came out - oh, let's see.
In fact, this particular experiment was repeated. Elliot verbalised the process saying "when three went away two came out". He then counted each column of three cubes and said 10.
Teacher: How did you know 10?
Elliot: Because I took out one, two, three,...., 10 (touching each column in turn as he counted by ones).
Teacher: Mmm.
Elliot: Each three I counted as one.
Elliot then checked using the CopyCat to confirm his result.
The initial problem of finding F(2,3) of 30 pieces was presented again. The teacher told Elliot that the machine would work. His task was to decide how many pieces would come out. Elliot was also told that he could do anything he liked with the computer except input 30 items. Elliot chose to input 3, and observed that 2 items were output. He then focused on the 30 cubes which were still arranged in 10 columns of 3. He began counting each column by 2's:
Elliot: Two, four, six,..., 16,18,20. Twenty will come out.
Elliot confirmed this with the CopyCat.
On the following Tuesday, four days later, Shannon and Elliot visited for another session. The teacher reviewed the problem of finding F(2,3) of 30 with Elliot while Shannon was assigned another task. Elliot remembered immediately that 20 pieces was the output. He also gave a clear explanation:
Teacher: Can you tell why it could be 20?
Elliot: Because um, say you put 30 in right? The machine it puts um, it puts 3 in, takes 3 apart, and for each 3, 2 comes out. That's the answer.
Teacher: And how many 3's are there in 30?
Elliot: 10.
The contexts in which these children have been learning fractions - namely, an operator interpretation as exemplified in the functioning of the CopyCat machine - have highlighted a discontinuity between unit fractions, for example, F(1,3), F(1,4), and F(1,5), and their multiples such as F(2,3). and f(2,5). Elliot and several other of the children in the study have been able to deal with a variety of problems involving the fractions F(1,2) and F(1,3). But unlike other contexts commonly used to teach fractions, such as region and area representation, F(2,3) does not naturally develop from its related unit fraction.
It is relevant to ask what F(2,3) might have meant for Elliot as a structured unit - within the confines of the data as described. It is tempting to claim that initially F(2,3) had almost no meaning for Elliot, because he was not able to answer the question appropriately. But this is not accurate. What he said was 22. Where did 22 come from? The most likely interpretation is that Elliot, while not active in answering the prior

question of finding F(2,3) of 24, accepted Shannon's answer of 16, and extrapolated this to the next problem: 30 is six more than 24, and six more than 16 is 22. The fact that 22 was so ingrained for Elliot supports the interpretation offered. It was indeed quite a rational and logical answer for him! The sequence of experiences for Elliot leading to success was for him to understand the process the CopyCat used to find F(1,3) of a collection of items, followed by reconsideration of the problem of finding F(2,3) again. This was not easy for Elliot. He did not automatically notice the structure that the F(1,3) CopyCat exhibited as it processed the input items - specifically, disappearance of a set of three input items followed by the appearance of a single output item, repeated. Elliot also benefitted from observing the F(2,3) machine in action. This gave him the necessary assistance to transfer to the original problem.

The fact that Elliot was able to reproduce the solution four days later, as well as verbally explain the basis of the solution, indicates that knowledge had taken root in his mind. The fraction F(2,3) had begun to exist as a scheme, the basis of which for Elliot was the action unit of two for three. Still, however, it is not clear to us that Elliot yet sees F(2,3) as twice f(1,3).

References

Changeux, J. P. (1985) Neuronal Man. The Biology of Mind. Translated by L. Garey. Oxford: Oxford University Press. pp. 163 - 166, 167.

Cobb, P., Wood, T. and Yackel, E. (1991) A constructivist approach to second grade mathematics. In von Glasersfeld (ed.) Radical Constructivism in Mathematics Education., pp. 157-176. Dordrecht: Kluwer.

Davis, G.E. (1991) Fractions as operators and as cloning machines. In R. P. Hunting and G. Davis (eds.) Early Fraction Learning, Recent Research in Psychology, pp. 91 - 102. New York: Springer Verlag.

Edelman, G.R. (1987). Neural Darwinism. The Theory of Neuronal Group Selection. Basic Books: New York.Freudenthal, H. (1983) Didactical Phenomenology of Mathematical Structures. Mathematics Education Library. Dordrecht: D. Reidel.

Hart, K.M. Fractions. In K.M. Hart (ed.) Children’s Understanding of Mathematics: 11 - 16, pp. 66-81. London: John Murray.

Hunting, R. P., Davis, G. and Bigelow, J.C. (1991) Higher-order thinking in young children’s engagements with a fraction machine. In R. P. Hunting and G. Davis (eds.) Early Fraction Learning Recent Research in Psychology, pp. 73 - 90. New York: Springer Verlag.

Kieren, T. & Nelson, D. (1978) The operator construct of rational numbers in childhood and adolescence - an exploratory study. The Alberta Journal of Educational Research, 24, 1.

Kieren,T. and Southwell, B. (1979) The development in children and adolescents of the construct of rational numbers as operators.The Alberta Journal of Educational Research, 24, 4.

Lambek, J. (1966)Lectures on Rings and Modules. Massachusetts: Blaisdell

Steffe, L.P., von Glasersfeld, E., Richards, J. and Cobb, P.(1983) Children’s Counting Types: Philosophy, Theory and Applications. New York: Praeger Scientific

Steffe, L.P., Cobb, P. and von Glasersfeld, E. (1988) Young Children's Construction of Arithmetical Meanings and Strategies. New York: Springer Verlag.
von Glasersfeld, E. (1991) Introduction. In von Glasersfeld (ed.) Radical Constructivism in Mathematics Education, pp.xiii-xx. Dordrecht: Kluwer.
Meaning of fractions