Constructivism, concept maps and procedures: making meaning in school mathematics

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Introduction

This paper seeks to move part of the debate on constructivism in mathematics education to a pragmatic level, to that point where teachers have to move children from exploration and construction of meanings in mathematics to the reality of learning whole number algorithms. The significance of constructivism is a point of meaningful debate, and is a highly desirable activity in school mathematics, but when do children learn efficient algorithms? This paper looks at a range of research activities, some current, some reported previously.

The term constructivism is not used solely in mathematics education, for example, it is used in science education and indeed exists in its own right as a description of a way of learning or of knowing, of making meaning. One of the difficulties associated with the term then is that it has so many meanings (Cobb, Yackel and Wood, 1992; Ellerton and Clements, 1992; Pateman and Johnson, 1990), perhaps not as many meanings as there are people who use the term, but certainly many meanings each with its own nuances. On the whole there is one important characteristic accepted as central to all views on constructivism: that learners construct their own meanings (Ellerton and Clements, 1992; Sinclair, 1990). Exactly what this means, how it is achieved, how pedagogy assists the process and how we assess the effectiveness of all this is problematic. Steffe (1991) outlines ten principal goals for researchers in this field, and to some extent these goals provide principles for instruction, as do Underhill's assumptions about construcivist learning (1991). Even so, it is still the case that constructivism and pedagogies associated with it continue to be problematic: the literature and discussion are replete with polemic arguments. This controversy shows itself in spirited, and sometimes heated debates.

A recent paper which appears at first glance to have its ideological position well removed from that of constructivism (Hall, 1992), was concerned with the teacher's role in ensuring learners adopt efficient procedures in the solving of whole number algorithms. The present paper seeks to bring together the tenets of constructivism with the first mentioned paper's emphasis on learning efficient algorithms. In particular, the present paper explores links between the highly abstract and highly desirable goals of constructivism, especially with regard to the development of cognitive maps, and the making of meaning, and the more mundane, pragmatic, yet equally desirable goals of learners completing algorithms efficiently. This paper examines the role of procedural analogies in helping learners construct meanings about mathematical concepts and their written forms, in the context of constructivism.

Constructivism

Ideologies that I associate with mathematics education in schools include seeing learners as individuals, that learning is an active and reflective process, that learning cannot be given knowledge as in the common analogy of filling an empty jug. I believe learners have rights, so even if the 'fill-the-jug' model was plausible I would view it as unethical to subject learners held compulsory in classrooms to lesson after lesson of inactivity, rote learning and subjugation. Nor are learners mice or pigeons whose behaviour is modified to some predetermined uniformed blandness, conservativeness or conformity through the use of rewards and punishments.

There is considerable support for these views if one looks to a sufficiently wide range of literature, as I have noted elsewhere (Hall, 1978, 1982, 1985). Both my views and those of a range of constructivists can find support in teachers' and curriculum developers' interpretations of Piaget's work, in the child centred educational approaches of the Progressive Education movement, and in the range of educational values associated with names such as Rousseau, Froebel and Dewey. An important addition to this list of supporters is that of the influential Nuffield Mathematics Project dating from the 1960s where the idea of learning by doing was quintessential to the project's philosophy. Indeed the so-called 'old Chinese proverb'

I hear and I forget,
I see and I remember,
I do and I understand

was a statement of the Nuffield project's philosophy. Given the lack of research in this area, and the general lack of empirical evidence at the time, these supports were important to those seeking change in mathematics education. This support is still important today, especially when the back-to-the-basics movement passes by in its regular cycle of political expediency, shortsightedness, narrow minded accountability, and its ever elusive search for the quick-fix.

For me, constructivism is not a new idea. It may be a new label, it may be fashionable, and it may draw together more ideas than ever before, but large chunks of it are hardly new. In seeking pragmatic solutions to some of the ideologies listed above, especially concerning the need to teach for understanding, I regard much of my activity years ago as consistent with constructivism (Hall, 1978, 1981a, 1981b).

An important element in constructivism is the idea that mathematics is a social construction. This is sensible enough, in that school learning takes place in a social milieu, and that learning will be more efficient if discussion is encouraged. In particular, cooperation and social interaction increase the likelihood of cognitive conflict and so stimulate learning (Pateman and Johnson, 1990). The question as to whether mathematical knowledge exists independently of the learner, and if it does not, then learning must be a reconstruction of socially imbedded realities is a view that I'm still coming to terms with. I accept that mathematics knowledge is not value free, and so may not be the objective and pure study so often assumed. Mathematical knowledge may develop from social interaction (Cobb, Yackel and Wood, 1992), but even if it does, and given that individuals are likely to have ideosyncratic cognitive structures, no amount of empirical data on these matters will provide a unique model for instruction nor provide any inconsistency with teachers declaring large chunks of mathematical knowledge as important to learn.

Von Glasersfeld sees constructivism as a theory of knowing, radically different from traditional theories of knowing. For Von Glasersfeld knowledge is not an iconic representation of an external world 'but rather a mapping of ways of acting and thinking' (1990). It involves the notion of ownership: that is, by constructing knowledge the learner may believe 'I know this because I worked it out myself' (Ellerton and Clements, 1992). While I have strong support for the power of knowledge learnt for oneself, and believe that it is typically accompanied by confidence building, more effective cognitive structures, and the ability to correctly use the knowledge, these benefits are a long way from any logical necessity concerning ownership. If ownership is part of the solution to contemporary problems associated with mathematics education, then the range of meanings associated with ownership, and its unclear functions and benefits, are part of the problem. For example, the fact that a teacher has helped learners to construct their own knowledge, as opposed to telling them what to do and how to do it, says nothing about owning or

not owning this new knowledge. Are we trying to rewrite our cliche as

I hear and I forget,
I see and I remember,
I do and I understand,
I know and I own?

Beyond the questions of learning by doing, and the perhaps not very fruitful discussions on ownership, is the issue of what this knowledge looks like. If it is an essential element of effective mathematics education, and so of constructivism and of teaching for understanding, that the the learner develops a richly connected network of mathematical skills and concepts, what are we expecting this cognitive end-product to look like?

Concept Maps

One of the major objectives of school mathematics education for me, is the development within individual learners a richly connected cognitive network of mathematical concepts and skills (Skemp, 1989a, 1989b). That is, in coming to learn mathematics, whether it be through problem solving or through a more directed approach, as students gain meaning they are required to make a large number of assimilations and accommodations. Without this gaining of meaning, little in the way of significant mathematical education has occurred. Indeed Steffe (1990) talks of curriculum as the construction of 'a network of mathematical concepts and operations that could deepen, unify, and extend [learners'] conceptions of mathematics curriculu[ms]'. One of the many roles of the teacher in this process of helping learners develop richly connected cognitive networks is to encourage reflective thinking through the provision of a broad a range of pedagogical approaches and learning activities. Such variety will assist learners to gain as broad as possible an understanding of mathematical skills and concepts, to enable them to discuss these meanings proficiently, and to establish connections between related concepts.

Learning Procedures

One does not have to be a very experienced teacher of mathematics to know that existing cognitive structures, particularly as related to procedural aspects of mathematics, are often resistant to change. Indeed incorrect procedures do not always seem affected by remedial instruction - students regularly fall prey to 'silly' errors, combine partially remembered earlier methodologies almost at random or fall back on a better remembered but erroneous methodology.

To the general public, procedures may be easily learned, but we know that this learning is often incomplete and short lived. Teaching mathematical procedures is well known to be problematic, indeed some of the written processes we teach may not be the most appropriate (Mansfield, 1990), and some of our 'helpful hints' may actually interfere with learning (Bobis, 1992). And although students perform algorithms as we may want, their mathematical understandings and richness of cognitive networks may not be of the same order as that of their teacher.

Classrooms are complex places where at any given time an almost infinite set of variables are in action, with the teacher aware of only some, able to influence fewer and in control of very few. Stephens (1990) lists a range of variables influencing one important variable in the classroom, that of communication. Such influences range from historical traditions and constraints, through a range of affective variables related to teachers' and learners' values and beliefs about mathematics, and mathematics teaching and learning, to society's expectations and assessment techniques. It is fairly obvious then that no one theory is likely to be able to provide a definitive, unique instructional approach (Hall, 1983): there are simply too many variables, not only diverse and often out of the teacher's control, but subject to unpredictable change.

Perhaps the best we can hope to achieve is a description of variables most likely to encourage learning, expressing this in the context of a theory of instruction. In this way, those learners who want to learn will have their chances of learning increased. That is, I am suggesting, indeed I am stating as fact, that no learning theory or theory of instruction will assist those learners who do not want to learn, do not pay attention to the teacher, who spend little time on-task, lack motivation or see no value in schooling or education.

What then can researchers advise teachers about instructing those students who want to learn? One possibility is the procedural analogy theory developed by Ohlsson and Hall (1990) that describes the cognitive function of concrete materials in the teaching and learning of school mathematics.

Procedural Analogy

Large numbers of students gain little in the way of understandings from school mathematics, and the use of concrete embodiments to increase the likelihood of understanding has yet to be convincingly verified empirically. The construction of meaning, the development of mathematical skills and concepts into a richly connected cognitive network, requires the manipulation of mathematical representations: but manipulation alone may not result in the learning of those standardised written procedures that must continue to be an important goal of school mathematics. The procedural analogy theory described briefly here is a theory of instruction in arithmetic, and has its basis in both cognitive science and mathematics education. In addition to the original publication concerning this theory (Ohlsson and Hall, 1990), aspects of the theory have been presented elsewhere (Hall, 1990, 1991, 1992; Ohlsson, 1991). This theory is relevant to those settings where concrete materials are used to increase learners' arithmetic skills and understandings. The use of such materials as an attempt to allow operations on ideas that are essentially abstract is hardly a new idea, but the empirical data supporting such notions is inconclusive.

The raison d'etre for concrete materials is to assist learners to develop and internalise those mathematical concepts and skills represented by these materials, and operations on these materials. In cognitive science terminology, the materials assist the learning of declarative and procedural knowledge. The procedural analogy theory describes how concrete materials assist this declarative and procedural knowledge to be developed to the required target behaviour. Simplification, procedural analogy and symbolism lead finally to automatic responses.

The theory predicts that the pedagogical usefulness of an embodiment is a function of the degree of similarity of the procedure for the embodiment to the procedure for the target symbolic representation, and argues that this relationship can be quantified. That is, this theory describes specific ways in which teachers may use concrete materials so as to increase the effectiveness of intended learning outcomes, and argues that these various approaches can be quantified and compared.

Algorithms involving the four operations with whole numbers make up a significant portion of school children's mathematical experiences, certainly up to the beginning of secondary school. The completion of these algorithms is known to be problematic. We will all be familiar with the range of incorrect approaches and the inventiveness of learners in creating incorrect solutions to algorithms. For example, subtracting one number from another is deceptively simple, especially if you always take the smaller from the larger.

The basis of the procedural analogy theory rests on the use of concrete materials as an aid to assist learners in drawing parallels between the numbers represented by these materials and the numbers represented in an algorithm, and between arithmetic processes represented by operations on these materials, and those

steps leading to the solution to an algorithm. Further, the theory asserts that while these materials may be used in a wide range of ways to achieve a correct answer, there are some ways that will be more effective than others because they more closely mirror the desired target behaviour.

Table 1: Procedural Analogy

MAB procedure

0.0 82 - 53
  0.1 Subtract 263 from 5H, 4T, 2U
1.0 Process units
  1.1 Take 3U from 2U (cannot)
    1.1.1 Trade for more units
    1.1.2 Move 1T from 8T to bank, bring back 10U
    1.1.3 Join 10U and 2U
    1.1.4 Recall 10U + 2U = 12U
  1.2 Take 3U from 12U
  1.3 Recall 12U - 3U = 9U
  1.4 Record answer, 9U in answer space
2.0 Process tens

  2.1 Take 5T from 7T
  2.3 Recall 7T - 5T = 2T
  2.4 Record answer, 2T in answer space
3.0 Read answer (2T 9U)

Target procedure
0.0 82 - 53

1.0 Process units
    1.1 Take 3 from 2 (cannot)
        1.1.1 Trade for more units
        1.1.2 Recall 8 - 1 = 7
        1.1.3 Cross out 8, write 7
        1.1.4 Write 1 next to 2
        1.1.5 Recall this is 12
    1.2 Take 3 from 12
    1.3 Recall 12 - 3 = 9
    1.4 Record 9 in answer space
2.0 Process tens
    2.1 Take 5 from 7
    2.3 Recall 7 - 5 = 2
    2.4 Record 2 in answer space
3.0 Read answer (29)

The right hand side of Table 1 shows that for the novice there are 16 steps in the target procedure leading to the correct answer for this algorithm. Certainly over time chunking of steps would take place, as would the development of a more automatic response, so the actual number of steps is likely to decrease. On the other hand, for those learners whose recall of number facts is poor, and requires finger counting or the like, the initial number of steps will be increased. Kamii (1989) provides an excellent example of the complexity of a seemingly simple representation when she contrasts the 'simple' representation of 32 using groups of objects with the extremely complex mental structure that learners must impose on the objects, by giving it order and hierarchy. For novices then, simple subtraction algorithms are not simple, they are complex multi-step operations with numerous opportunities for error. And it is here that the value of the procedural analogy theory can be seen, both in drawing the teacher's attention to the need to cover every aspect of the algorithm to be taught and providing a meaningful sequence for learners, one leading to a correct answer. Table 1 also shows one use of Multibase Arithmetic Blocks (MAB) and the target algorithm that is developed from this material. The steps emphasised both in the use of MAB materials and in the target algorithm are not unique, and must be developed by the teacher. Once the teacher has decided on the target behaviour, an effective sequence can be developed for the concrete material. According to the procedural analogy theory the closer the analogy between the concrete procedure and the target procedure the more effective will be learning outcomes. The procedural analogy theory uses an isomorphism index ($I_{1,2}$) as a measure of analogy between the two procedures. The index is given by the formula

$$(N_1 + N_2 - 2) - (D_1 + D_2)$$

$$I_{1,2} = N_1 + N_2 - 2$$

where N1 is the number of steps in the first procedure, N2 the number of steps in the second procedure, D1 the number of steps in the first procedure but not in the second, and D2 the number in the second procedure but not in the first. In Table 1, N1 = 16, N2 = 16, D1 = 2 and D2 = 2 giving a high isomorphism index of 0.87. Slight variations in the steps will lead to a lower isomorphism index. The theory maintains that the higher the I1,2 value the more effective will be the value of the concrete materials, and the greater the level of learning outcomes. That is, the procedural analogy theory allows an analysis of teaching

steps prior to teaching and provides a method of measuring likely pedagogical success.

My research seeks to verify this procedural analogy theory, but my present purpose is more concerned with the value of this theory in helping learners make mathematical meanings. That is, to what extent will learners establish meaningful procedures for solving whole number algorithms through actions on concrete embodiments, where high levels of analogies are emphasised?

Meaning from Procedural Analogy

It could be argued that there are ideological difficulties with this procedural theory: it appears to be more about procedures to obtain the correct answer than about constructing meanings. But this is an unwarranted oversimplification. Table 2 outlines one effective approach to the use of concrete materials for developing procedures in solving algorithms: it shows that the first steps are where construction of mathematical ideas takes place. That is, if these steps are avoided or rushed then understanding will be made more difficult and will likely not be achieved by a large proportion of learners. For many learners, the first steps outlined are essential in establishing meaning.

Table 2: Algorithm development
with concrete materials

Free play
Using materials in any manner to achieve correct answer
Using materials in a prescribed manner to achieve correct answer
(Using materials, writing corresponding expanded algorithm)
Using materials, writing corresponding contracted algorithm
(Algorithm only, check with materials)
(Algorithm only, place value language)
Algorithm only, face value language

The procedural analogy theory is applicable to the final steps shown in Table 2. That is, if it is necessary for learners to eventually adopt standardised forms of algorithms, the procedural analogy theory provides a tool for assisting instruction to move from the use of concrete materials to establish meanings to the development of algorithms where these meanings are represented in formal, written procedures. This position is not inconsistent with or in opposition to constructivism, where the goal is still that of students' learning correct mathematical understandings. If the procedural analogy theory is implemented by the teacher so as to allow students to construct meanings through taken-as-shared practices, and allows both teachers and learners to be active interpreters making sense of their world, then a constructivist approach is still being used (Cobb, Yackel and Wood, 1992).

This paper began with a statement of ideological position, and support for the notion of constructivism. While the range of meanings of constructivism was considered a difficulty, it was none the less acknowledged that constructivism is an important goal in mathematics education. I have argued though that standardised algorithms are an important goal in school mathematics, that this goal will be better achieved through the application the procedural analogy theory outlined in the paper, and that this goal is not inconsistent with constructivism.

Many research questions ensue from this paper. How generalisable is the procedural analogy theory? Does application of this procedural analogy theory encourage learners to develop a richly connected network of cognitive structures? Indeed there are many questions about the procedural analogy theory, about pedagogy, about meaning and about cognitive structures. The answers to such questions may help verify the procedural analogy theory, and may provide a pedagogical approach supporting both the values of constructivism and the goal of learners being able to solve algorithms efficiently.

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