A Metacognitive, Conceptual, Rule-based, Problem-solving Approach to Teaching Maths

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A metacognitive, conceptual, rule-based, problem-solving approach to teaching maths called “Advance and Justify” has been trialled with classes at The Dunedin Institute for Learning for 12 months with self assessed poor achieving in maths long term unemployed and young un-employed who have no scholastic achievement. Results demonstrated major gains in accuracy, understanding, and confidence. Previous research studies, two at Intermediate school level, and two with trainee teachers in the Kura Kaupapa Maori Teacher Training Course at the Auckland College of Education demonstrated similar major and enduring gains in accuracy, understanding, and confidence.

Every major educational system gives mathematics an important place in the curriculum. This importance is justified because mathematics is indispensable to commerce, engineering, science and technology, and because it is useful to the individual in their everyday activities. Despite this recognition of importance and despite intensive efforts over the last few decades we still, on a world wide basis, see many students doing poorly in maths (Skemp, 1987).

There are many varied reasons for this poor achievement. A growing list of researchers believe that a major contributor to this pattern of poor performance is our over-emphasis on correct answer rather than on mathematical process (Kruetetskii, 1976; Schoenfield, 1979; Resnick, 1983; Polya, 1984; Gagne, 1984; Whitney, 1985; Skemp, 1987). Emphasis in teaching must now be placed on understanding mathematical processes involved in problem solving and making maths relevant to the learner.

A teaching method developed in New Zealand called “Advance and Justify”, based on a metacognitive (teaching the learner to think about why they are doing what they are doing, and why they are thinking the way they are), conceptual, rule-based, problem-solving approach to learning mathematics, using a seven step problem solving strategy and only a few basic rules (Appendix A), has been successful in improving confidence, making maths relevant and increasing understanding in three separate studies (Firestone, 1988; Firestone 1989a; Firestone, 1990a,Firestone 1990b; Firestone, Nepe, Graham & Glynn 1990, 1991), and shows promising results in two studies still in progress: 4) an inter-subject study of adult students who have poor academic achievement and long histories of unemployment pre/posttesting academic achievement in tasks including computation, fractions and algebra; 5) a comparison study of experimental and control standard 2-4 primary school classes from four different schools comparing academic achievement in the tasks: number, addition, subtraction, multiplication and division.

The teaching method incorporates concepts and ideas which are not new, but have been cited by educational psychologists and cognitive psychologists for many years. These ideas have included viewing Mathematics as a vehicle for improving reasoning, logic, and organisation of thought (Schultz, 1927); appreciating that development of knowledge occurs in two phases: the automatic unconscious acquisition followed by gradual increases in active, conscious control over that knowledge (Vygotsky,1962); the human mind is a pattern-making and pattern-using system (De Bono, 1976); appreciating that problem solving requires five phases of reflective thinking (Dewey, 1933); appreciating that thinking ability is developed only through appropriate activity (Krutetskii, 1976); using mathematical knowledge in problem solving to lead to a general attitude of "common sense" or "using your head" (Polya, 1965); the importance of training in cognitive and metacognitive skills for successful problem solving (Brophy, 1986; Lester, 1978, 1985).

The success of the teaching method “Advance and Justify” in building confidence and making maths relevant, lies in the students accepting that mathematics can be used as a vehicle to learn how to better:

1) recognize problem types,
2) use knowledge,
3) synthesis knowledge

4) break problems down into manageable parts,
5) generalize their knowledge,
6) use meaningful rules in an organized logical manner to solve problems
7) use these problem-solving skills in solving their everyday problems.

“Advance and Justify” emphasises students' understanding of the nature of the problems they are working with, as well as understanding their own attempts to apply known rules to advance towards a solution. The strategy focuses closely on assessing understanding of mathematical knowledge as well as accuracy of outcomes. Students are encouraged to ask and answer a series of questions which facilitate their recognition of the problem type (not just a particular problem), their selection of some knowledge or rule from similar or related problems, their justifying of their use of that knowledge or rule in the present context, and their evaluation of how effective their use of that knowledge or rule has been. Students' demonstration of their appropriate application of these thinking processes is at least as important as their obtaining correct problem solutions. Research has shown “that procedures must take on meaning and make sense or they are unlikely to be used in any situation that is at all different from the exact ones in which they were taught” Resnick (1983, p 478).

The problem-solving strategies incorporated in “Advance and Justify” stress recognition and understanding of conceptually similar processes and ideas across deceptively different domains of application. For example, addition of fractions, addition of algebraic terms, addition of geometric angles, addition of integers, and addition of time all employ the same logic and the same generic rules. Understanding the process should allow the operation of addition to be applied to any of these examples. Unfortunately, many teaching programmes destroy this powerful commonality. Separate and distinct algorithms are "taught" to handle each example. Successful application of each algorithm does not guarantee sufficient understanding to transfer the operation from one type of example to another. Opportunities for discovery of generic rules and mathematical concepts are lost.

The teaching method has been demonstrated with abstract problems in basic arithmetic, fractions and algebra. Problems, grouped together in similar domains called tasks, were selected which facilitated the generalization of ideas between tasks (see Appendix C for tasks and sample problems for each study). This logical progression increases the learners understanding about maths and helps them learn to transfer their ideas and knowledge to other domains. Their ability to use concepts and basic procedures increases because they are building larger and more powerful schemata. The learners' over-all picture of mathematics gets bigger and more useful as they tie together the understandings of the different tasks by using the same rules and concepts in different ways. The importance that rules have in learning how to think is clearly given by Bloom (1976): rules are probably the major organizing factor, and quite possibly the primary one, in intellectual functioning. The S Æ R connection, once proposed as the unit of mental organization, has now been virtually replaced by the rule in the theoretical formulations of many psychologists. [Even they] ... are forced to concede that the proponderance of observed human behaviour occuring in natural situations is rule-governed (p.129).

The tasks algebra and fractions were chosen to demonstrate the power of the teaching method "Advance and Justify" because these two topics are frequently considered difficult to teach by teachers, and difficult to learn by students. This problem domain is a valuable place to begin since the knowledge is well defined and self contained, and more importantly, may provide a supportive framework for learning general problem solving skills that can generalize to many other problem domains. Problems of the type:

have the ideal properties for learning problem solving. "They require subgoals and a general principle has to be applied to obtain the subgoal Greeno (1980, p 9)

Results of Previous Experimental Studies:

Study One

A pilot study with Form 1 participants from an Intermediate School in Dunedin, compared three groups of low achievers (n = 6) in maths, using a multiple base line design (Firestone, 1988). The study was of six weeks duration, consisting of thirty, 50 minute lessons. The three groups achieved excellent results on understanding over 9 tasks involving complex algebra and fractions (see Appendix C) . They improved their pretest scores by more than 41 percentage points on task 3, by more than 52 percentage points on tasks 8, 2, and 7, by more than 66 percentage points on tasks 9, 6, 5, and 4 and by more than 84 percentage points on task 1 (see Appendix B). A comparison of tabulated results from pre and post questionnaire replies and interviews, indicated that subjects confidence, understanding, and liking for maths, significantly improved.

Study Two

The mathematical performance of Form 1 participants from two Intermediate Schools in Dunedin were compared. Five experimental Form 1 classes (n = 163), taught by a mixture of teachers (three volunteer relieving teachers with self assessed low confidence in teaching maths, one regular classroom teacher, and the researcher), with six, Form 1 classes (n = 178) taught in a conventional manner by their regular classroom teachers (Firestone, 1989a). This study, consisting of seventeen, 35 minute lessons, was of shorter duration than study one, and studied three fewer tasks (Appendix C). Comparing the percentage point increase of the experimental groups post-test over pre-test understanding scores, with the control groups percentage point increase of post-test over pre-test understanding scores, the experimental group achieved significantly larger percentage point increases (see Appendix B). Based only on measures of understanding, results suggest the poorest achieving experimental class out-performed the highest achieving class in the control school (see Appendix B, Summary of Statistical Restults). On the six fractions and algebra tasks, the experimental group achieved more than a 22 percentage point increase over the control group on task 2, better than a 35 percentage point increase on tasks 5 and 3, and better than a 47 percentage point increase on tasks 6, 4, and 1. A comparison of tabulated results from pre and post questionnaire replies and interview of the experimental group indicated the following: Subject's confidence, understanding and liking for maths, significantly improved (Appendix B). To the question "What do you think about using rules to solve maths problems", 68% of subjects responses were highly positive (Helped a lot; made it clear; really helped me; understand now; super; great) and another 24% gave a positive response (helped a bit; understand a bit better; alright; ok; ok but hard). To the question "How did you feel about maths before the maths program? How do you feel about maths now?", 62% of subjects gave a highly positive response (Hated maths before, now I like it; found maths boring, now I like it; didn't understand maths before, now I like it; didn't like maths before, now I like it; didn't have any confidence before, now it's easy), 9% responded positively (like maths a bit better; improved a bit; understand maths a bit better), 21% liked maths before the program and still like it, and 8% continued to dislike maths.

Teachers who believed they had low confidence in their ability to teach mathematics participated as experimental teachers in the second (1989) study in order to assess whether the teaching method is suitable for a majority of teachers, and whether the training objectives are quick and easy to learn. The training program for teachers with low confidence consisted of 24 hours of classroom instruction (12 hours of lecture and 12 hours of supervised problem-solving activity), and homework which required reading, All You Need to Know About Fractions and Simple Algebraic Equations (Firestone, 1989c) and working all the maths problems in Mathematics Problem Book: Fractions and Simple Algebra (Firestone, 1989b). The confident teacher had four hours of instruction, and also did the required homework. The experimental teachers response to interview questions at the end of the program indicated that they significantly improved their confidence and liking for teaching maths, felt that children would not have difficulty learning fractions or algebra using the methods of "Advance and Justify", observed a positive change in their pupils confidence and attitude towards the problem domain, felt that the training

objectives were not too hard to master, and felt all teachers would benefit from, and should have, this training program.

The control school teachers and pupils used their regular textbook, Modern School Mathematics Structure and Use (Duncan, Capps, Dolciani, Quast, and Zweng, 1980), and the Hawkes Bay Math Series (Hawkes Bay Maths , 1986) to cover the 12 topics detailed in the method section below. The control pupils were taught conventional lessons by their usual teachers. The pupils each had a copy of the Mathematics Problem Book: Fractions and Simple Algebra (Firestone, 1989b) and worked through the practice problems for the twelve topics.

The experimental school teachers and pupils used, All You Need to Know About Fractions and Simple Algebraic Equations (Firestone, 1989c) instead of their usual textbooks. Each pupil had a copy of the same problem book as the control school, and worked the same practice problems.

Study Three

An inter-subject (n = 11) study compared: pre-tests and post-test accuracy and understanding scores, pre-questionnaire and post-questionnaire tabulations, and questionnaire responses of participating teacher trainees from the Kura Kaupapa Maori teacher training program of the Auckland College of Education. The program was of two weeks duration, and consisted of 20 hours of lecture, and 16 hours of supervised problem-solving study where questions could be answered. For homework, required reading was, All You Need to Know About Fractions and Simple Algebraic Equations (Firestone, 1989c) and working all the maths problems in Mathematics Problem Book: Fractions and Simple Algebra (Firestone, 1989b). Results indicated a significant improvement in measures of confidence across all tasks, and a significant improvement in measures of accuracy and understanding when pretest and posttest results were compared (see Appendix B). In pre-questionnaire tabulations (see Appendix D for example of Questionnaire) the mean response for all 24 questions regarding confidence on a seven point scale (1 Very Confident, 7 Not Confident), was 5, with only 4 of the 24 responses indicating less than 3. In contrast the post-questionnaire tabulations showed the mean response for all tabulations was less than 3, with 21 of the 24 responses less than 3, and 14 of the 24 responses less than 2. Of the fourteen tasks involving fractions and algebra in study 3 (Appendix C), on measures of accuracy, there was more than a 45 percentage point increase of posttest over pretest on tasks 14, 11, 4, and 1, more than a 62 point increase on tasks 6, 5, and 7, more than a 47 percentage point increase on tasks 2, 13, and 8, more than a 83 percentage point increase on task 12, and more than a 70 point increase, on tasks 9 and 10 (Appendix B) . On measures of understanding, there was more than a 48 percentage point increase of posttest over pretest on task 1, more than a 58 percentage point increase on task 6 , more than a 65 percentage point increase,on tasks 5 and 3, more than a 73 percentage point increase on tasks 7, 2, and 14 ,more than a 81 percentage point increase on tasks 11, 13 and 12 , and more than a 93 percentage point increase on tasks 10, 9, and 8 (see Appendix B).

The fourth study which was also inter-subject (n = 12) compares pre-tests and post-test accuracy and understanding scores, and pre-questionnaire and post-questionnaire tabulations of the same participating teacher trainees as study 3. The study uses the same rules, concepts, and basic procedures as the previous study, and develops and adds a few more. The study starts by analyzing two tasks, multiplication and division, which the students had mastered during their primary schooling years. These tasks are usually learned in a rote manner, with little over-all understanding of how the knowledge used in multiplication and division could be built upon and used in other more complicated tasks. The study included the solution of problems involving: power of ten notation, algebraic multiplication (quadratics), square roots, whole and fractional powers of ten, and three types of exponential equations (see Appendix C). The program was of eight days duration, and consisted of 16 hours of lecture, and 16 hours of supervised problem-solving study where questions could be answered. Preliminary data is very promising. On measures of accuracy students improved their outcome of post-test over pre-test on all the hierarchial tasks, achieving improvement in excess of 40 percentile points on tasks 4 and 11, in excess of 56 points in tasks 3 and 5, in excess of 76 points in tasks 12 and 6, and in excess of 88 points for tasks 9, 7, and 8 (see graph in Appendix B).

Summary of Results:

For a quick over-all view of the results of the previous studies it is best to consult the graphs of accuracy, understanding and attitude in Appendix B.

1989 Study:

The Summary of Statistical Data table suggests that when ranked classes are compared the lowest achieving experimental class out-performed the highest achieving control class (67.5% over-all vs 64% over-all). The experimental class teacher was a volunteer person who had given up teaching and had assessed herself prior to the program as a teacher with very low confidence in teaching maths. The control class teacher was the regular class teacher (for the purpose of confidentiality, control class teachers are not identified, but included teachers of self assessed high confidence in teaching maths). When a comparison was made between the results of a basic maths skills test (problems taken from their standard three and four textbooks) and the post-test results of the experimental maths work, we find that the experimental classes found the fractions and algebra work significantly easier than their regular work, Mean +24.7% points (all of the DMp–s were positive). For the control classes we find the opposite effect. They found the fractions and algebra significantly more difficult than their regular work, Mean –12.5% (all of the DMp–s were negative). An over-all difference between the two schools of 37.2 percentage points. When we compare the improvement of outcome between the pre-test and post-test results of the poorest achieving experimental class and the poorest achieving control classes (DMp–pre), we have a 53.7% increase vs a 6.1% increase.

The Graph of Pre/Post-test Paired Results suggest that the control classes approached the addition tasks as independent tasks with inconsistent transfer of knowledge from one task to another as the tasks became more complex. This result was in contrast to that observed in the experimental classes.

1990 Study 3 and 1990 Study 4 (PRELIMINARY DATA ONLY) :

In both 1990 studies, a comparison of the graphs of pre-test and post-test results show the significant achievement made in both accuracy and understanding. These results are all the more significant when we consider that they were achieved by the students over two-two week training periods . The Graphs of Attitude indicate that the students attitudes towards the tasks significantly improved over the training period.

Conclusion:

The plea to use mathematics as a vehicle for improving reasoning, logic, organization of thought, and metacognitive skills is an old one. It was eloquently expressed by Schultz (1927):

No other subject suffers so much and becomes so valueless as mathematics, when treated by mechanical modes of study (p. 13) ... The principal value of mathematical study arises from the fact that it exercises the reasoning power more, and claims from the memory less, than any other secondary school subject. The study of mathematics should result in the development of power, rather than the acquisition of facts. Not he who knows a great many mathematical facts is a good mathematician, but he who can apply these facts intelligently (p. 18).

Nearly sixty years later we hear Brophy (1986) making a similar plea to educators:

students achieve success consistently by beginning at their level, moving in small steps, and preparing them sufficiently for each new step so that they can adjust to it without much confusion or frustration. . . . students will need cognitive and metacognitive training, (a) setting and committing themselves to appropriate goals, (b) monitoring and appraising their performance as they work, and (c) reinforcing themselves for goal attainment. (Brophy, 1986, p. 341)

The skills which "Advance and Justify" uses and teaches to improve reasoning, logic, organization of thought, and metacognitive skills, are:

Learning to integrate knowledge of the problem (e.g. students ask themselves; "What is the problem telling me about itself?"; "What type of problem am I dealing with?"; "What is the problem asking me to do?").

Relating the problem to previous knowledge (e.g. students ask themselves; "What do I know about this type of problem?"; "What rules or concepts will I need?").

Learning to focus on goal (e.g. students ask themselves; "What do I know about the answer, in what way does the present state differ from the goal state?").

Learning about problem structure and process (e.g. students ask themselves; "How can I advance towards the answer?" "Did this Advance Help?"; What else can I do?").

Using problem solving techniques (e.g. sub-goaling, backward thinking, forward thinking, means end analysis, substitution, analogy, diagrams, problem representation).
Developing metacognitive skills (e.g. students ask themselves; "What rule or knowledge do I have which justifies this advance?"; Why do I think I have the correct answer?" "Why did I use this particular rule? “Why have I done what I've done?").
Developing Transfer (e.g. "What rule or knowledge do I have which justifies this advance?"; Why do I think I have the correct answer?"; "Why did I use this particular rule, why have I done what I've done?"; "Have I used this rule or idea before, how useful is it?"; "What other things could I use this idea or rule for?").

The ideas used in "Advance and Justify" are not new. They can be found in any modern text on Cognitive Psychology (e.g. Matlin, 1989). There are teaching methods which incorporate many of these ideas (e.g. Dewey's five phases of reflective thinking; Polya's four phases; Lester's Method; and methods based on: Osmosis, Memorization, Imitation, Cooperation, and reflection, all cited in Firestone, 1990a, pp. 13-16). What makes "Advance and Justify" effective is the simplicity of the package. It uses only a few but very powerful and generizable rules, and a simple seven step problem solving strategy (Appendix A). When we compare the poor results that New Zealand students achieved by international standards with the results achieved in these studies it suggests the importance of support for further research on trials in the ecologicaly valid setting of a whole school environment, and on further research for investigating the effectiveness of the teaching method “Advance and Justify” in other domains.

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