COGNITIVE FUNCTIONING IN MATHEMATICAL PROBLEM SOLVING DURING EARLY ADOLESCENCE

Kevin F. Collis, Jane M. Watson and K. Jennifer Campbell

University of Tasmania
GPO Box 252C
Hobart, Tasmania 7001

(A paper prepared for AARE Annual Conference, Gold Coast, 1991)

The authors wish to acknowledge the support of the ARC in carrying out the research reported in this study (ARC. Ref. No. AC9031914). Our thanks also go to Ms Susan Johnson whose able assistance, especially in relation to interviewing and data analysis, is gratefully acknowledged.

ABSTRACT

Problem-solving in school mathematics has traditionally been considered as belonging only to the mode of thinking concerned with making logical connections between data and the mathematical model and then teasing out the relationship between the variable in the model and the concrete symbolic mode. Little, if any, attention has been given to the place of the intuitive processes of the ikonic mode at this level.

This project has set out to explore the interface between logical and intuitive processes in the context of mathematical problem-solving with a view to designing guidelines for teaching and assessment procedures. The paper will present the results obtained in the early stages of the study.

Introduction

Problem-solving in school mathematics has traditionally been considered as belonging only to the mode of thinking concerned with making logical connections between data and the mathematical model and then teasing out the relationship between the variable in the model and the concrete symbolic mode. Little, if any, attention has been given to the place of the intuitive processes of the ikonic mode at this level despite the well-known use of these processes in the thinking of researchers at a high level of science and mathematics (Hadamard, 1954). Moreover, research in out of school thinking (Resnick, 1987) during the past decade points to the predominance of intuitive mode strategies in everyday problem-solving especially in mathematical situations with unschooled individuals. Now, however, educators have realised the limitations of the traditional approach and the mathematics curricula which are appearing throughout the world as a result of national enquiries (Cockcroft, 1982 U.K.; National Council of Teachers of Mathematics, 1989 U.S.A.) are focusing on getting a balance between the two modes of functioning. The problem is that there is little relevant research which deals with learning or its assessment from this point of view (there is of course an enormous amount of literature dealing with assessment in the concrete symbolic mode). The result is that curriculum, teaching and assessment decisions are being made on the basis of what is recognised to be good exemplary practice at this stage (e.g. California State Department of Education, 1989; deLange, 1987; Graded Assessment in Mathematics, 1988). A recent analysis (Collis & Romberg, in press) of some assessment items from one of the projects (California State Department of Education, 1989) shows clearly how deficient and inefficient this procedure is likely to be even in the relatively short term.

Even a cursory search of the literature will make clear why there has been this reliance upon good practice and teachers' intuitions as the model for assessment in this area of problem solving. Most of the research concentrates on school-based problem solving during the very early elementary areas or on teasing out variables associated with the various processes in the psychological domain that appear relevant. In addition a series of studies describing the nature of 'outside school' or 'work place' mathematical problem solving have been reported in the past decade (Carraher, 1989; Resnick, 1987; Scribner, 1986). It is fair to say that although many of these studies have figured in the intuitions of mathematics educators when designing recent problem solving assessment procedures and items there has been little effort to look at the problem with a specifically assessment mode in mind. The study reported here is an attempt to redress this balance and to look at the problem solving processes of the students in high school with a view to devising a rational means of assessing their competence.

Theoretical Orientation of the Study

The study reported here is grounded in the Biggs & Collis formulation of the development of cognition as updated in 1991 (Biggs & Collis, 1991) from its original 1982 version. In addition the work of Collis and Romberg (in press) on the assessment of open-ended items in mathematical problem solving has been utilized in the basic framework for gathering the empirical data. Let us examine briefly these two contributions to the background of this study.

Biggs & Collis (1991) and Collis & Biggs (1991) point out that much of our thinking in the area of problem-solving is multimodal. In particular, students at the early high school level, when presented with a novel mathematical problem, have available three modes of intellectual functioning which they can bring to bear to seek a solution, sensori motor, ikonic and concrete symbolic. The first of these is usually not of great significance in school-based problem solving and will not be discussed here. The second is highly developed by the time students reach the age level of interest here and has reached this high level largely without school help. The third is very dependent on school-based teaching, and the subsequent learning, of the concrete symbolic systems of reading, writing and arithmetic - in this paper the last is the major concern.

Biggs & Collis (1991) argue following Piaget and others that the ikonic mode of functioning begins in early childhood (i.e. around 18 months) with the beginning of true thought in the form of the internalisation of action (Piaget, 1950), a form of imaging which Bruner (1964a, 1964b) later referred to as an "ikon". From a very rudimentary form externalised in the one word "sentence" of infants it generalises, throughout the pre-school years with the aid of language, to the richly imaged world of the early school child. Throughout its development, ikonic thought not only draws on imaging but very often has heavy over-tones of the child's affective life. Using this mode for encoding the real world young children find much satisfaction in interpreting many of the interactions they see in real life in terms of myths and stories (Egan 1984).

This then is the beginning of the ikonic mode but it continues to develop until well into adulthood where ikonic thought clearly goes beyond primitive explanations to the intuitive thinking displayed in such areas as aesthetics, mathematics and science. The latter two domains in particular appear to make regular use of this mode in problem solving; Hadamard, 1954, supplies many examples from mathematics and Kekule's use of this mode in solving the problem associated with the structure of the organic ring is well known.

The ikonic mode is clearly not simply to be associated with the presymbolic thought of early childhood but continues to grow in both power and complexity into adulthood. Moreover, as the later modes develop, there appears to be an interaction between modes which enables the typical ikonic mode strategies to be applied to symbolic ways of representing reality.

The move to the concrete symbolic mode marks a significant shift from a direct imaging of reality to a written, higher order symbolization of reality. These second order symbols form a system with direct referents in the experienced world. The symbols also have a logic and an order between themselves which are represented directly in reality and give us one of our most powerful tools for acting on the environment. These symbol systems include written language, mathematics, maps, musical notations and so on which are by no means the result of natural learning in the way ikonic or sensori motor achievements may be regarded as natural. They are the products of deliberate, carefully organised instruction. Indeed, the mastering of these systems and their application to the real world activities of the individual are most often considered as the primary task for schooling during the compulsory years as they form the basis for much of the every-day reasoning and record keeping in our modern adult society.

From this starting point Collis and Romberg (in press), analysed a sample of the kinds of open-ended problem solving items used in the variety of recent assessment projects listed above. The results of the analysis of students' responses and, in some cases, instructors' expectations (e.g.California State Department Department of Education Project, 1989) led to the hypothesis that students had some common approaches to school based mathematical problem solving and that in general the instructors' expectations were not compatible with these. The former is of major concern here.

Collis and Romberg (in press) found that all of the projects that they analysed in their study made use of open questions. These consisted of items in which the students had to construct their own responses. A wide variety of this item type is currently being tested throughout the world and, although the items vary on many dimensions, they all set out to test the higher order aims of mathematical problem solving and to reveal the student's reasoning as he or she moves towards a solution. The variations between the items includes: time allowed for solution, from traditional examination type conditions to projects that might take a semester to complete; degrees of cooperation and outside help allowed, from none to unlimited; amount of mathematization of the data required; amount of writing required, from virtually none to persuasive essays. As would be expected, each has its own particular advantages and disadvantages and most projects reported problems with achieving objectivity.

Although there was a variety of techniques used to set up the problem situation, the basic format for this type of item came down to one in which the context was set by a series of propositional statements followed by questions to which the student(s) was expected to construct a response. The ways in which the context set for the problem could be varied and the open-ended nature of the construction required of the response made different demands upon both student and assessor from the traditional form of test item. Let us consider a student's task when faced with a typical item of this type.

The Student's Task

As was described above, the problem is placed in context by a series of propositional statements, followed by a question which seeks a response. The student needs to take the given propositions and decide on a course of action which might be schematized as follows (adapted from, Collis and Romberg, in press):

After absorbing the data an initial decision is made and the student proceeds down column L or R; at row C, Column L splits and leads in the case of C(i) to an irrelevant conclusion (in the given context) or in the case of C(ii) to an intuitive relevant solution. The route shown on Column R is associated with traditional mathematical problem solving. However, it is rare that a student will stay purely on one track (L(ii) or R) if the problem is novel. In this case, there is likely to be movement both ways at either rows B or C or both --this will apply whether the subject is basically following route L or route R. The work place mathematics literature shows that minimally educated adults solving problems in the ikonic mode utilize any number skills that they have (Carraher, 1989) while the use mathematically competent individuals make of the ikonic mode is well known (Hadamard, 1954). It would appear that the task for the instructor is to find ways of assessing what the individual is doing in rows B and C.

Testing at CL is concerned with the ability to work cooperatively, to handle a particular context, to use outside resources, to define data to be used, to judge degree of precision required, to assess attitude and intuitive ability in problem solving, and to judge flexibility of reasoning. Testing at CR is focused on knowledge and understanding of mathematical procedures, mathematical concepts, communicative ability with mathematical language, and problem solving with mathematical models.

CL and CR together seem to define recent ideas of what is required of students in mathematical problem solving. Under the traditional assessment procedures the focus was on CR abilities in a very narrow range; tests for the new generation of problem solving skills will need to be devised so that the assessor can look at both CL and CR and their interaction.

This project was set up as the first step in devising guidelines for designing assessment techniques for testing students' skills in handling this new generation of problem types. Before guidelines were formulated it was necessary to examine empirically the hypotheses implied above which were developed from theory and which formed the basis of the earlier analytical study by Collis & Romberg (in press).

In context of the data gathering proposed this is most easily considered in relation to the following questions:

Question 1. Do students have the same primary strategy by which to approach different types of problems? If not, then do students have the same primary strategy by which to approach particular types of problems?

Question 2. Overall, are the strategies used dependent upon the obvious characteristics of the item?

Question 3. In the process of solving each type of problem, what typical path (see Figure 1) do students follow, if any?

Question 4. Do specific individual characteristics, as assessed by standard measures of imaging and creativity, relate to the nature of students' strategies?

The Study

Preliminary Considerations: Crucial to examining the questions were, (1) the selection of the items which would form the main instrument for gathering the data, (2) devising suitable interview procedures to obtain relevant data and (3) determining the appropriate form of analysis which would be suitable to answer the questions using the data obtained while taking into account the necessarily small sample of students which we could afford to involve.

The Items:

It was necessary to select problems which, if possible, could be solved by techniques associated with the different courses of action a student might decide to take (see Figure 1). For example, the following problem can be solved in more than one way.

Hungry Men Problem

Three tired and hungry men had a bag of apples. When they were asleep one of them awoke , ate 1/3 of the apples and went back to sleep. Later a second man awoke, ate 1/3 of the remaining apples, and went back to sleep. Finally, the third man awoke and ate 1/3 of the remaining apples, leaving 8 apples in the bag. How many apples were in the bag originally? Explain your solution.

As Watson (1988) has shown the most likely methods of solution involve symbols and diagrams; that is, methods that are primarily school-taught, based on concrete symbolism and traditionally associated with mathematical problem solving in the school context. It was adopted as a model for items for which the students were likely to be attracted towards a concrete symbolic approach.

An example of a second type of problem, where a different set of solution methods would be expected, is the following:

Cube Painting Problem

A cube that is 3cm by 3cm by 3cm was dipped in a bucket of red paint so that all of the outside was covered with paint. After the paint dried, the cube was cut into 27 smaller cubes, each measuring 1cm on each edge. Some of the smaller cubes had paint on 3 faces, some on 2 faces, some on only 1 face, and some had no paint on them at all. Without drawing the cube, explain how you would find out how many of each kind of smaller cube there are.

This problem would appear to involve a significant visual or imaging, intuitive component. This is so even if the individual attempts to draw a diagram or picture during the course of seeking a solution - it certainly did not lend itself to any concrete symbolic, school based approach. This item was adopted as the model for later items which were meant to suggest the use of the ikonic mode as the primary strategy.

In summary, two categories of problems were identified according to expected solution strategy, those which would suggest the use of concrete-symbolic processes (CS) and those which would suggest ikonic processing

(IK). On this basis, two sets of problems were prepared for use in interviews (see Appendices A and B) with both types of problem present in each set. Two pairs of problems in each set were intended to have the same structure and thus were isomorphic in the sense used by Greer (1991).

Data Collection:

Sample

Two Advanced Mathematics classes in a Tasmanian High School were selected as the sample. There was a total of thirty-eight students altogether, nineteen in year 9 (mean age: 14 years 10 months, 11 boys, 8 girls) and nineteen in year 10 (mean age: 15 years 10 months, 12 boys, 7 girls).

Imaging Tests

At the first testing session three group tests were administered, two to assess aspects of each student's ability to image situations and one to obtain a measure of each student's creativity. All thirty-eight students completed this battery of tests which consisted of the following individual tests: the Betts Questionnaire on Mental Imagery (QMI) (Sheehan, 1967), the Gordon Test of Visual Imagery Control (Gordon, 1949) and the well-known 'Uses of a Brick' Creativity test (see Guilford, 1967). This set of tests was selected to provide data to assist in answering Question 4.

The Interviews

As it was not possible to interview all thirty-eight students in the time available sixteen were selected for interviewing individually on their problem solving strategies in relation to the item types described above, eight students were allocated to each problem set of 5 problems.
For each interview, the interviewer sat next to the student. Problems, or parts of problems, were presented to the student one at a time on white cards. Once the problem was presented students could start solving the problem whenever they were ready; each had a pen and paper, ruler and calculator available if required. Half of the students received the problems in the order in which they appear in Appendices A and B, and the other half in reverse order. In addition, to check on the effect of images associated with each part of a question on the method of solution, some problems were serialised and presented to students as follows (see Appendices A & B): first, the question (E), followed by the other parts of the problem (A to D) in random order.

Students were asked to tell the interviewer everything that came into their minds while reading the problem (or a part of the problem) and while solving it. During the course of the interview, the interviewer prompted the student, if necessary and where appropriate, by asking if they 'saw' any aspects of the 'story', if they were anticipating what the problem was about or what information they thought they still needed to solve the problem. The interviews were tape-recorded.

Plan of Analysis:

The Interview Data, Categorisation:

If the student's interview responses were to be analysed in relation to the strategies being sought, a way of recording what the students were saying in terms of the mode (CS or IK) which each individual statement apparently represented was required. A variation of Haylock's Think-Board (Haylock, 1984) seemed to offer promise for exploring the responses in this way. Haylock developed the Think-Board, not as a research tool, but as a device to encourage the problem solver to consider different representations for a problem in attempting to find a solution. It was decided to adapt the Think-Board technique so that it could be used to record the problem solver's actual 'moves' during the process.

The adapted Think-Board appears in Figure 2. The broken line divides the board into two regions, the upper region represents Concrete Symbolic (CS) responses while the lower half represents Ikonic (IK) responses. Each of these regions is divided into three smaller regions by the diagonals, these smaller regions represent subsets of the CS and IK regions of which they form a part.

Starting at the top left, we have the following definitions for each of the smaller regions:

Structure Recognition: The recognition of a CS structure in a problem as similar to a previous example, eg, "This problem will be solved using the same mathematical ideas that I used in the last one."

Symbols: The recognition of an appropriate CS procedure to solve the problem, eg, "This should be able to be solved using simultaneous equations."

Diagrams: The use of a CS type diagram, eg,.Venn Diagram, Graph, etc to solve the problem.

Images: Reporting visual images related to the problem, eg, "I can see the pieces of the cube with one side painted."

Reality, Beliefs etc: The use of real world experience etc which appears to have some 'practical' relationship to the problem, eg, "Everyone knows that you can't run a car without petrol!"

'Aha' Experience: A sudden, apparently unbidden, 'insight', into the structure of the problem, usually visual, eg, "Oh I see what I have to do" or "I see what it's getting at."

Interviews were transcribed and each student's responses to each question were mapped onto a Think-Board. These maps were then transformed to paths in which the sections of the Think-Board were denoted as follows: Structure Recognition (S1), Symbols (S2), Diagrams (S3), Images (I1), Beliefs, Common-Sense, Reality (I2), and 'Aha' (I3).

The number of responses falling in the upper (CS) half of the board and the number falling in the lower (IK) half of the board for each question were determined and used as the basis for the statistical analyses. Where a response was seen to be a combination of more than one section of the Think Board it was decided that it should contribute to the score for each of those sections. An illustration of the procedure is included in the Appendix C.

Statistical Analyses Proposed:

To examine the questions that had been set it was deemed necessary to use techniques that would give some assurance that any differences that could be detected were unlikely to be the result of a chance association and thus it was thought appropriate to find some suitable statistical procedures to analyse the data. The procedures available, however, were obviously constrained by the small numbers and the nature of the data gathered. A summary of the statistical procedures used to shed light on each question follows.

Question 1. The interest here was whether the individual students did in fact adapt his/her strategy (basically CS or basically IK) when faced with items which were designed by the experimenters to favour one approach over the other. This was done in two steps. First, the data was examined overall to see if there was a consistent approach to all the problems regardless of type. Second, the data on each type of item was taken separately to see if there was a consistent approach within item type. The Cochran Q test (Siegal, 1956) was decided on as the appropriate statistical technique. Given this type of data and size of sample, it is a method of testing whether the probabilities of a given type of response are different in various conditions. The entries required for the tables were obtained by determining the primary strategy each student used for solving each problem. This was done by counting up the two types of response, if there were more IK responses than CS responses, a score of "1" was assigned; if not, the score assigned was "0". This data was then tabulated (see Table 2, Appendix D) and Cochran Q tests were performed on both sets of data.

Question 2. This Question was not concerned with the individual student strategies as such but used the pooled data to examine the more general notion that the students' approach to a problem in general would be determined by the obvious characteristics of the item. To do this, the total number of CS responses was counted as was the total number of IK responses and entered as appropriate against each of the two categories of problem. The results were entered into a two ¥ two table (see Table 3, Appendix D) and the likelihood of the result being attributable to chance checked by means of a chi-squared frequency test.

Question 3. This Question sought to check the students' movement between the modes as they proceeded towards a solution. A count was made of the number of times that each student changed from one mode to the other. Totals for each question were calculated by summing the number of times each student changed modes on that question. The total number of changes was 65 for the CS-type problems and 65 for the IK-type problems, giving means of 21.7 and 32.5. A dependent t-test was performed on these means to determine if the mean number of times students changed modes when answering CS-type problems was significantly different from the mean number of times students changed modes when answering IK-type problems (see Table 8, Appendix D).

Question 4. The final Question was asked in order to assess the effect of students' ability to create images on the nature of their strategies. The standard measures of imagery and creativity mentioned earlier were scored, producing four variables for each student. The total score on the Betts QMI yielded a vividness of imagery (Vividness) rating as well as a visual subscale which was included as a separate variable (VisVivid). A control of visual imagery score (Control) was provided by the Gordon Test of Visual Imagery Control. Finally, the fluency, flexibility and uniqueness scores on the Uses of a Brick Creativity test were summed to produce a measure of Creativity. The students interviewed were divided into High and Low groups on all four variables based upon a mean split (see Table 1, Appendix D). The problem solving strategies of the High and Low groups on each of the four variables were then compared by summing IK and CS responses of the students in each group and applying a chi-squared frequency test.

Results

The results of data analyses are presented separately for each Question.

Question 1. Do students use the same primary strategy (CS or IK) by which to approach different types of problems? If not, do students have the same primary strategy by which to approach particular types of problem?

Cochran Q tests revealed that overall ten problems, the primary strategies used by the students were not consistent across the the two different types of problems (CS & IK) ($Q = 38.86$, $p < .001$). This general finding held for each set of 5 problems considered separately ($Q = 26.52$, $p < .001$; $Q = 27.30$, $p < .001$).

A further analysis to check whether primarily IK strategies were applied in response to CS problems produced non-significant results,($Q = 3.00$, $p < .10$). Inspection revealed that the strategies applied in these cases were primarily CS (see Table 2, Appendix D).

However it was found that students consistently applied primarily IK strategies to IK problems generally ($Q = 0.00$, $p < .99$).

These results seem to indicate that there is not a predeliction on the part of the individual student for a specific strategic approach to problem solving, rather it is the nature of the problem itself which is the basic factor in determining the strategy to be used.

Question 2. Overall, are the strategies used dependent upon the obvious characteristics of the item?

Table 3 (Appendix D) shows the 2 x 2 table set up to test the hypothesis. The chi-squared analysis of the pooled CS and IK responses of the whole sample for all the items was significant (chi-sq = 79.15, p <.001) and support the view that a priori CS problems elicit more CS responses than a priori IK problems and vice versa (see Table 3, Appendix D). These results support the conclusions from Question 1 that it is the apparent nature of the problem itself that determines the strategy to be used.

Question 3. In the process of solving each type of problem, what typical path do students follow, if any?

The dependent t-test on the means of the number of mode changes in each category of question produced a significant result (t = 2.86, p <.02), indicating that students tended to change modes significantly more often when answering CS-type problems than when answering IK-type problems. A check of the transcripts suggests that when students are working on a CS approach, and get stuck, they feel free to use their imagination to make progress. Whereas when they use their imagination/intuition to solve the problem they are typically satisfied with the result, be it correct or incorrect, and see no point in supporting it with mathematics. The exception to this occurs when they need to use basic number skills, such as counting or the four arithmetical operations, to obtain a required number as the answer.

Question 4. Do specific individual characteristics, as assessed by standard measures of imaging and creativity, relate to the nature of students' strategies?

Summary tables for the chi-squared frequency tests related to this final question appear in Appendix D (Tables 4 to 7). The results indicated that students who are high on overall Vividness use IK strategies more frequently than students who are low on Vividness (chi-sq = 10.00, p <.01). Since Vividness is a general measure of vividness of imagery which includes such components as kinaesthetic, tactile, and aural imagery, among others, the visual subscale (VisVivid) was treated as a separate variable. The results of the chi-squared test in this case gave similar results, and it was found that students with high VisVivid scores are more likely to give IK responses than their counterparts with low VisVivid scores (chi-sq = 12.27, p <.001). Students with high Creativity scores tend to use IK strategies more than students with low Creativity scores (chi-sq = 5.82, p <.02), regardless of the type of problem. Students with high Control scores in relation to visual imagery, are no different from students with low

Control scores on either category of item (chi-sq = .72, p <.50).

Conclusion:

Within the limitations set by the data and the small sample, this study highlights some important factors which must be taken into account in the teaching and learning of problem solving, in novel situations, in school mathematics.

1. There seem to be two basic approaches to problem solving at this level, one based on CS, school taught procedures, the other related to ikonic mode processing which latter, in this study, means appeals to common sense, everyday life and visualisation and the use of intuitive (aha!) experiences. It is interesting to note that students at this level, who see what they believe to be a solution using ikonic approaches, see little point in backing up their solution by an appeal to mathematics.

2. In the course of solving a problem, students will move from mode to mode but are more likely to move from CS to IK when a school based method fails to satisfy rather than vice versa. They do however, move both ways, in the case of IK to CS procedures (or knowledge), after the problem is really solved.

3. The characteristics of the problem and its setting appear to determine the basic strategy.

4. Students who have the ability to obtain clear visual representations of a problem and have high levels of creativity in developing ideas are most likely to be comfortable with IK approaches.

Some Educational Implications:

1. Assessors of mathematical problem solving at this level, should take into account the fact that the obvious problem characteristics are going to determine the basic strategy of the students.

2. Teachers should be aware of the significance of the IK mode in problem solving and adjust their mathematics teaching accordingly.

References:

Biggs, J.B., & Collis, K.F. (1991). Developmental learning and the quality of intelligent behaviour. In H. Rowe (Ed.), Intelligence: Reconceptualization and measurement. Melbourne: ACER.

Bruner, J.S. (1964). The course of cognitive growth. American Psychologist. 19, 1-15.

California State Department of Education (1989). A question of thinking: A first look at students' performance on open-ended questions in mathematics.

Sacramento: Author.

Carraher, T.N. (1989). Numeracy without schooling. In Actes de la 13eme Conference Internationale I. Paris, France: PME.

Cockcroft, W.H. (1982). Mathematics counts: Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. London: Her Majesty's Stationery Office.

Collis, K.F., & Biggs, J.B. (1983). Matriculation, degree requirements, and cognitive demands in universities and CAEs. Australian Journal of Education, 27, 41-51.

deLange, J. (1987). Mathematics insight and meaning. Utrecht: OWOC.
Egan, K. (1984). Educational development. Oxford: Oxford University Press.

Gordon, R. (1949). An investigation into some of the factors that favour the formation of stereotyped images. British Journal of Psychology, 39, 156-167.

Graded Assessment in Mathematics (1988). London: Macmillan Educational Ltd.

Greer B. & Harel S. (1991). Visualisation in Problem.Solving. Working Group, PME Conference, Assisi July 1991.

Guilford, J.P. (1967). The Nature of Human Intelligence. New York: McGraw Hill.

Hadamard, J. (1954). The Psychology of Invention in the Mathematical Field. Princeton, NJ: Princeton University Press

Haylock, D. (1984). A mathematical think-board. Mathematics Teaching, 108, 4-5.

National Council of Teachers of Mathematics (1989). Curriculum and evaluation standards for school mathematrics. Reston, VA: Author.

Piaget, J. (1950). The Psychology of intelligence. London: Routledge & Kegan Paul.

Resnick, L.B. (1987). Learning in school and out. Educational Researchers, 16(9), 13-20.

Scribner, S. (1986). Thinking in action: Some characteristics of practical thought. In R.J. Sternberg and R.K. Wagner (Eds.), Practical Intelligence. Cambridge: Cambridge University Press.

Sheehan, P.W. (1967). A shortened form of Betts questionnaire upon mental imagery. Journal of Clinical Psychology, 23, 386-389.

Wason, P.C. and Johnson-Laird, P.N. (1972). Psychology of Reasoning: Structure and Content. Cambridge, Mass: Harvard University Press.

Watson, J.M. (1988). Three hungry men and strategies for problem solving. For the Learning of Mathematics, 8, 20-26.

Please address requests for figures to the first author at the Centre for Education, University of Tasmania, GPO Box 252C, Hobart, Tasmania, 7001. Facsimile: (002) 202569.

APPENDIX A: Problem Set One

Hungry Men Problem
Three tired and hungry men had a bag of apples. When they were asleep one of them awoke, ate 1/4 of the apples and went back to sleep. Later a second man awoke, ate 1/2 of the remaining apples and went back to sleep. Finally, the third man awoke and ate 2/3 of the remaining apples, leaving 2 apples in the bag. How many apples were there originally?

Doctors and Dentists Problem
A journalist looking for a story came across the following two survey results. The Dentists' Association found that half of the population is nervous about going to the dentist. Independently the Medical Association found that half of the population is nervous about going to the doctor. Michael read the article and thought everyone must be afraid going to either a dentist or a doctor. Is Michael right? Why or why not?

Cube Painting Problem
A cube that is 3cm by 3cm by 3cm was dipped in a bucket of red paint so that all of the outside was covered with paint. After the paint dried, the cube was cut into 27 smaller cubes, each measuring 1cm on each edge. Some of the smaller cubes had paint on 3 faces, some on 2 faces, some on only 1 face, and some had no paint on them at all. Without drawing the cube, explain how you would find out how many of each kind of smaller cube there are.

Waitresses Problem
A. Three waitresses, Julie, Alice, and Tina, put all their tips in one jar.
B. Julie went home first and took 1/3 of the money as her share.
C. Alice, not knowing that Julie had taken her share, took 1/3 of what was there as her share.
D. Tina, unaware that the others had already taken what they thought were their shares, took 1/3 of the remaining money.
E. There was $8.00 left in the jar. How much did the waitresses have in tips in the beginning?

Fuel-savers Problem

A. Three companies have designed fuel-savers for car engines.
B. Company X's fuel-saver will reduce fuel consumption by 20% ,
C. Company Y's fuel-saver will save 30%,
D. and Company Z's saves 50%.
E. Caroline thinks that if she uses all three fuel-savers on the engine of her car she won't need to buy any more petrol. Is Caroline right? Why or why not?

APPENDIX B: Problem Set Two

Envelopes Problem (Wason and Johnson-Laird, 1972)
Subject presented with set of 4 envelopes:
Unsealed , Sealed , 30c stamp , 43c stamp
Subjects told that there is a general postal regulation:
If a letter is sealed, then it has a 43c stamp on it.
Subject is to select those letters that would definitely need to be turned over to find out whether the regulation was being violated.

Chickens and Goats Problem
A farmer is counting the chickens and goats in his yard. He counts a total of 50 heads and 140 feet. How many chickens and how many goats does the farmer have?

Trip Problem (Wason and Johnson-Laird, 1972)
Subject presented with set of 4 cards: Bus, Car), Oatlands, Launceston
Told each card has a town on one side and a mode of transport on the other.
Given rule:
If I go to Launceston, then I travel by bus
Subject is to select those cards that would definitely need to be turned over to find out whether the rule was true.

Garage Sale Problem
A. A lady buys 20 plates at a garage sale. Some are large plates and some are small.
B. The small plates have a pattern with two roses on them.
C. The pattern on the large plates has four roses.
D. She counts 56 roses altogether on her plates.
E. How many large plates and how many small plates does the lady have?

Wason's 4-card Problem (Wason and Johnson-Laird, 1972)
Subject told each card has a letter on one side and a number on the other.
Given rule:
If a card has a vowel on one side, then it has an even number on the other side.and cards: E K 4 7
Subject is to select those cards that would definitely need to be turned over to find out whether the rule was true.

APPENDIX C: Examples of Scoring Procedure

KEY: Structure Recognition (S1), Symbols (S2), Diagrams (S3), Images

(I1), Beliefs, Common-Sense, Reality (I2), and 'Aha' (I3).

Example 1: Cube Problem
I am imagining a whole cube that isn't cut up. (I1)
I can see the cube being cut up. (I1)
Now I can see the cube in slices coming away from each other and I can see how many have 2, and how many have 3... (I1)
It is like a Rubix cube. (I2)
[From this image the student worked out: 3 sides (I1)
and no sides (I1)
and 2 sides. (I1)]
Then I added the numbers together and found that there were 6 left so there are 6 with one side painted. (S2)

For this student the path is as follows: I1, I1, I1, I2, I1, I1, I1, S2.
and the resulting scores are: IK 7, CS 1.

Example 2: Chickens and Goats Problem
There's 50 altogether. (S2)
If there were 20 chickens, that takes away 40 of the feet, which leaves 100 divided by 4 is 25, so that's 20 plus 25 which doesn't add up so it doesn't work. (S2)
If 50 feet were on the chickens that would be 25 chickens, and then there'd be 90 left.... divided by 4 won't work. (S2)
I don't know if there's another way of doing it. If I divided it all by 2 so they'd all be chickens there'd be 70 pairs.... 40.... 80.... 120. (S2)
I don't know, I've done one like this before but I don't know how I did it. (I2)
I know, divide it into 3 bits because 1 is a chicken and 1 is half a goat and 1 is the other half of the goat. But it is 50 so I don't know. (S2)
Okay, 5 heads and 14 feet. (S1)
I take off 10 gives 5 heads, 14 feet and then I'll change it back. (S2)
[Draws 5 heads with 2 feet each. Counts. Gives 2 apiece of the the remaining 4 to 2 heads.] (S3 and S2)
[Refers to diagram.] I have 20 goats and 30 chickens . (S2)

For this student the path is as follows: S2 S2 S2 S2 I2 S2 S1 S2 S3 S2 S2
and the resulting scores are: IK 1, CS 10.

APPENDIX D: Analysis Tables

Table 1. Summary Statistics for Imaging and Creativity Measures (n = 16).

Variable	Mean	Standard Deviation	Range
Vividness	94.31	23.38	53 - 144
VisVivid	12.67	3.57	7 - 19
Control	19.13	4.60	8 - 24
Creativity	17.56	5.57	7 - 25

Table 2. Primarily IK (1) or CS (0) Responses to Problems by Students.

Obs.	Responses to CS Problems 1	2	CS Row Totals L	L2	Responses to IK Problems 1	2	3	IK Row Totals L	L2	Row Totals All problems L	L2
1	0	0	0	0	1	1	0	2	4	2	4
2	0	0	0	0	0	1	0	1	1	1	1
3	0	0	0	0	1	0	0	1	1	1	1
4	0	0	0	0	1	1	1	3	9	3	9
5	0	1	1	1	1	1	1	3	9	4	16
6	0	0	0	0	0	0	1	1	1	1	1
7	0	0	0	0	0	1	1	2	4	2	4
8	0	1	1	1	1	1	1	3	9	4	16
9	0	0	0	0	1	0	1	2	4	2	4
10	0	0	0	0	1	1	1	3	9	3	9
11	0	0	0	0	1	1	1	3	9	3	9
12	0	0	0	0	1	1	1	3	9	3	9
13	0	0	0	0	1	1	1	3	9	3	9
14	0	0	0	0	1	1	1	3	9	3	9
15	0	0	0	0	1	1	1	3	9	3	9
16	0	1	1	1	1	1	1	3	9	4	16
Col. Totals	0	3	3	3	13	13	13	39	105	42	126

Table 3. Problem Category and Response: Observed and Expected* Frequencies.

	CS-type Problems	IK-type Problems
CS Response	191 (139.28)	83 (134.72)
IK Response	84 (135.72)	183 (131.28)

Table 4. Vividness and Response: Observed and Expected* Frequencies.

	High Vividness (n=10)	Low Vividness (n=6)
CS Response	141 (159.2)	134 (115.8)
IK Response	174 (155.8)	95 (113.2)

Table 5. VisVivid and Response: Observed and Expected* Frequencies.

	High VisVivid (n=8)	Low VisVivid (n=8)
CS Response	111 (131.4)	164 (143.6)
IK Response	149 (128.6)	120 (140.4)

Table 6. Creativity and Response: Observed and Expected* Frequencies.

	High Creativity (n=9)	Low Creativity (n=7)
CS Response	136 (150)	139 (125)
IK Response	161 (147)	108 (122)

Table 7. Control and Response: Observed and Expected* Frequencies.

	High Control (n=8)	Low Control (n=8)
CS Response	131 (136)	144 (139)
IK Response	138 (133)	131 (136)

* Expected frequencies appear in parentheses.

Table 8. Dependent t-test on the Mean Number of Modal Changes for CS and IK Problems.

Degrees of freedom	Mean number of modal changes on CS problems	Mean number of modal changes on IK problems	t-value	Probability (2-tailed)
15	32.5	21.7	2.855	.012