COGNITIVE AND AFFECTIVE ASPECTS OF MATHEMATICAL PROBLEM SOLVING - A CASE STUDY

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In the resolution of mathematical problems, the whole individual is involved. It is impossible to compartmentalise the cognitive aspects separately from the affective aspects. A researcher may attempt to examine the different domains separately, but what is observed as an affective feature relates closely to and exerts influence over the cognitive features of the situation and vice versa.

In order to gain some insights into the various factors which contribute to success and failure in mathematical problem solving, the writer undertook a project to investigate the problem solving behaviour of teacher education students, children and other adults.

The variables to be investigated were isolated as a result of an examination of the situation existing in the School of Education at the University of Western Sydney, Nepean, and the recent introduction of the new N.S.W. K-6 Mathematics Syllabus. The latter places considerable emphasis on the importance of problem solving and mathematical thinking in all aspects of mathematics. The situation in the School of Education is such that a significant percentage of the teacher education students are mature aged women without recent backgrounds in mathematics for the task which will be theirs as teachers in primary schools.

These two circumstances led to a consideration of the following variables:

The availability and use of concrete materials in mathematical problem solving Personality characteristics of the problem solvers Anxiety experienced by the problem solvers Motivation Beliefs Perceptions Language Mathematical background Problem solving strategies Reasoning skills

These variables were examined through the use of a stimulated recall technique whereby subjects are videotaped as they solve problems or perform problem solving tasks and are immediately stimulated to recall their feelings and the processes involved by viewing the videos. An audiotape is kept of the material recalled.

Instrument

In order to make the situation as little threatening as possible for the subjects, it was decided that the problems to be solved should be relatively easy, non-routine problems. They were selected from Stacey and Southwell (1983). Where appropriate, materials such as grid paper, counters, matchsticks, and paper were available for the student to use in the task set.

Subjects

The subjects were three students in the third year of their BEd (Primary) course. The three subjects were mature aged women with a mixed history in mathematics. The three of them had been unsuccessful in school mathematics but were highly motivated to improve. They were, however, very anxious about the subject and about their skill in teaching it.

Two of the subjects had attended the mathematics workshop which was provided for them before their entry into the pre-service course and had found it very useful in helping to reduce their initial anxiety and to become familiar with the university as a whole. Nevertheless, although reasonably successful in the mathematics education subjects taken in the two and a half years since, they were still fairly insecure about their ability to work independently on

mathematical tasks. One of them asked whether it was possible to work on the set tasks in groups because she felt more secure that way. One made the comment when she volunteered that she felt that she was willing to try anything that might help her to improve her mathematics. As this study might fall into that category, she was willing to volunteer.

All three are very hard working, had generally satisfactory academic and teaching records and are very co-operative.

Design

It was decided to use a stimulated recall procedure so that each subject might be prompted to recall her thought processes and feelings as she worked at the problems. In this procedure the student works on a problem and the situation is video-taped using two cameras, one which faces the subject and the other which looks over the shoulder of the subject, thus capturing the written and manipulative work of the subject. Immediately following the completion of the task, the video is played while the subject watches and an interviewer, in this case the writer, stops the tape intermittently and asks the subject about the video. Typical questions are "What were you thinking at that point?" or "How did you feel at that point?" "What made you do that?" etc. The intention is to trace the thought processes and feelings of the subjects as they worked on the problems. The interview after the actual task was audiotaped for later analysis.

This procedure requires immediate recall upon completion of the task and therefore was considered to be more able to yield the exact processes and feelings used by the subject than other methods where delays may take place. The immediate replay of the video helps to prompt the subject as she reflects on the sight of her own body language and written work. This immediate replay also contributes to the reliability of the process.

It is recognised that there are some inherent difficulties with this method. One is the possibility that subjects may report not just in terms of their actual experience but include the results of their reflection as they view the tape. This phenomenon was observed in the case of one of the subjects but was quite obvious to the writer. This particular difficulty would be less evident with children who would not necessarily be able to distinguish between current and past thoughts.

Analysis

Analysis is carried out by viewing the videotapes and listening to the audiotapes and observing the physical, verbal and diagrammatic cues given by the subject in the total process of first solving the problem and then of recalling the experience.

To establish validity, a colleague also viewed the tapes and consulted concerning the writer's conclusions. The tapes were also transcribed for further analysis and validation.

Timing

The problem solving sessions were carried out over the period of five months, the first few being considered as very exploratory to see if the design was feasible.

RESULTS

i) Availability and Use of Concrete Materials

Subject J. had concrete material available in six out of the seven problem solving sessions. She chose not to use the paper provided for Quick and Toastie and expressed a desire to have counters as well as grid paper for the Milk

Crate question.

Subjects B. and P. were provided with materials and both used them though not always productively.

ii) Anxiety

A high level of anxiety was obvious on the part of all three subjects. This was indicated by a number of physical cues such as looking from the question to the written work and back again in order to make sure the written work is in keeping with the question, shuffling in the seat whenever an obstacle seems to have arisen, a grim face, looking at the researcher every now and again to gain confirmation of whether she is on the right track and frequent giggling.

Other verbal cues came from the audiotape and from the observation of the video. For instance two of the subjects commented on the fact that they had to count at various stages to make sure they had the right number or were obtaining the right number of objects. Another verbal cue came from one subject who said that she was looking for tricks in the question because she feels she is easily misled.

iii) Motivation

All three subjects indicated high motivation to succeed in mathematics in order to be able to teach mathematics well. This high motivation was indicated by such statements as: "I could sit for hours and do questions like this." "I enjoy mathematics as long as I have time." "I never feel I can't do it - it's just that I need the time." Even the subject who said, "I hate the question simply because it is maths", in fact was prepared to sit and work at the question. "It's not that I'm giving up," she said, "but getting away from it for a while." Then the same subject expressed feelings of pain when faced with a mathematical problem. She indicated she had no recollection of her primary school mathematics and associated high school mathematics with a great deal of suffering and pain which has carried over to situations now she is faced with what she considers a mathematical problem to solve.

iv) Beliefs

At times throughout the problem solving sessions and the subsequent interviews, subjects made statements which indicated that they held certain beliefs about mathematics as a discipline and about themselves as mathematical problem solvers. One or two remarks also indicated beliefs concerning their personal situations.

(a) Beliefs about mathematics

In referring to Quick and Toastie, Subject J. said:"I didn't see it as a maths problem. It doesn't fluster me like maths." In a similar vein was the statement made in relation to the algebraic question. "I hated the question simply because it has numbers because it represents maths per se." Another statement which seems to reveal a certain belief that mathematics has no intrinsic beauty or challenge was: "Milkmen don't do that. This is the real world. They would not be worried about the even number in each row." A further statement which seems to contradict the previous one was: "I come from a large family and this would not be causing any problem at all." This was in relation to Sandwiches. The comment made in relation to Milk Crate "It is a pain, you know - the pain of trying to work something out. That's how I remember Maths. I can't remember primary school much but high school that is how I felt" is a comment indicating a certain belief concerning both mathematics as a subject and herself as a mathematician.

(b) Beliefs about Selves as Mathematical Problem Solvers

The statement by Subject J. "I remember my mind was being pre-occupied - the fact that I had done this before." seems to indicate a belief that if she had

done this before, she should be able to remember the solution.

"Good! I'm familiar with vending machines and very familiar with small change." seems to indicate a belief that as she was familiar with some of the situational elements in the questions then she would be able to reach a solution.

Subject P. stated that she is "always much more comfortable writing about maths in words than in symbols. (Pause) Probably as a child I got my symbols wrong and didn't get across my message. I always feel happy if I can write something out."

vi) Perceptions

The subjects' perception of the problem influenced their attitudes and methods of approaching the problem. As stated above Subject J. tried to recall past experiences because she perceived the question to be the same as one attempted before. This same act of remembering caused her to feel negative and anxious about the question. Perceptions also contributed a negative effect where the situation involved had previous connotations for the subject. When the problem was perceived as easy, however, as in the case of the Plant Growth, the subject moved reasonably quickly to a satisfactory resolution.

vii) Language

Language played a very direct role in the achievement of the subjects. This was evident from the initial reading of the problem and the need for clarification of some of the situations involved.

Subject P.'s reliance on words rather than symbols was related to the need for clarification.

viii) Mathematical Background

Subject J does not consider herself to be very bright and does not remember much about primary mathematics. She sat for the School Certificate at the Advanced level at the end of Year 10 and gained an ordinary pass. She claims never to have understood algebra though her sister's boyfriend helped her gain some understanding. She went to a girls' school where no one liked maths teachers and no one told her she was capable to doing mathematics. Her mother, however, did have much more faith in her ability. She completed her nurse's training in 1968 and did well. She recognises that mathematics is needed in nursing.

Subject B came from England at the age of 11 years and 9 months after having been introduced to arithmetic, algebra and geometry. She had no bad feelings about mathematics but later felt more inadequate. She loved algebra and geometry because drawing was usually involved. She left school at 14 1/2 years and thinks that mathematics is her best subject now. She likes the new K-6 syllabus and wants to help children develop their critical thinking skills.

Subject P recalls being an average student in the primary school though she did well until she encountered algebra. She did very poorly in algebra and resorted to girlish behaviour. Having completed a course similar to the N.S.W. Mathematics in Society course, she went into a basic algebra and calculus course at university where she managed to cope. She had, however, a poor image of mathematics. She did well in English and passed Physics and Chemistry because of her father's coaching and memorising formulae. She had little understanding. Her father encouraged his sons to do mathematics, not his daughter.

ix) Problem Solving Strategies

The strategies used by the subjects were categorised under three headings,

those related to:

1. the first stage in solving problems, characterised by Polya as Understanding the Problem;

2. the second and third stages, i.e. Deciding on a solution strategy and carrying it out; and

3. the fourth stage; i.e. Review.

In understanding the problem, subjects relied mainly on reading the question and looking for key words. If materials were available, they were used to help clarify the situation, but if they were not available, the subjects tended to make assumptions and use diagrams in their solution strategy.

The use of key words was justified by Subject J. in terms of the following: "Numbers and words give the clues".

Subject P. read the questions very carefully but appeared not to be convinced that she knew what they were about. She preferred to write something. "The first inclination was to record in language - writing about it - use language so it's exactly clear to myself." Later in the interview the same subject said "I'm always much more comfortable writing about maths in words than in symbols.

The main strategies used by Subject J. in tackling the problems were logical reasoning and recall. The strength of the urge to remember past experiences or suggestions made by the writer in lectures is alarmingly high. Attempts to reason were evident in four of the seven problems attempted. Enumerating all possibilities, looking for a pattern, sub-problems and trial and error were other strategies used. It was noted that the subject tried more than one method in all but the algebraic question.

Subject J. reviewed her solution in only three of the seven problems attempted. The strategy used in two cases was to write down what she had found. In the latter case, this occurred during the solution process rather than at the end of it. The other indications of reflection taking place were given by statements like "On paper you can refer back", "I kept looking back", "Better to get away from it for a while", and "I do that when I'm trying to sort things out. It is just ... really just stopping to think - breathing to help me concentrate."

Subject B. used a number of strategies in working one problem, including the use of concrete material, logical reasoning, and exploring all possibilities. She tended to plunge straight into the question without querying its meaning after her initial reading. She did reflect on her solution strategy as indicated by the following statement: "I was thinking of coming from the corners and building square numbers which I don't think would work anyway. I think I was still fixed on having a block but I knew I had to move some separately but couldn't figure out how."

Subject P. tended to use one strategy only and, perhaps due to anxiety, needed to clarify the questions attempted before going very far. She preferred to write ideas down in words rather than symbols so tried to reason through to a solution even while manipulating materials. She constantly stopped to reflect on what she was doing.

x) Reasoning Skills

As indicated above, the subjects all displayed attempts to use reasoning skills. Statements made which deserve consideration are as follows:

Subject J

"I just took any number which was 73 and thought, oh well, the difference is supposed to be 18 and 73 makes 7 and 3 make 10, so I'll just subtract 18 and

see what the result is and it did turn out that they both fitted the category of totalling 10 and having a difference of 18 so I continued to see how many things I could come up with."

"You could never have at any one time more than one 20 cent piece and 10 cent pieces could come into combinations either 3 tens, 2 tens, 1 ten and 5 could be any combination that fitted in with the rest or could be all 5s."

Subject B.

"I was thinking of the 18 in relation to the 24 and then I thought of what numbers - the factors of 18 and the number of rows and columns - I was sort of trying to work out in my mind even numbers which is why thought we mightn't be able to do it because I wasn't thinking of odd numbers. I was thinking of, you know, multiplying something by something to get 18 or something of that nature rather than adding up numbers to make 18."

Subject P.

"What I was doing there - I read the question to say it was more important to say how much money I could have and still not be able to make 20c change and I worked out you could have any amount and not make the 20c, change as long as you limited what you could do with the number of small coins."

DISCUSSION

1. Use of Concrete Materials

The use of concrete materials was a important factor in helping the subjects produce satisfactory resolutions to the problems. However, the use of concrete materials did not necessarily guarantee a satisfactory resolution. The materials were used in three main ways. In the initial stages of approaching the problem, materials were used to clarify the intention of the question. In the solution stage, materials were used to work towards the resolution in a logical manner. Throughout the problem solving process, materials were used as a prop to bolster self-esteem and limit anxiety. In this latter aspect, success was not always evident in either the problem solving or in limiting anxiety.

A slight conflict between the use of concrete material and the role of writing in problem solving emerged in one interview. The subject concerned made a statement to the effect that she needed to write down what she had done with the materials in order to have a permanent record on which to look back. Mason et al (1982) make the point that if we record the process we follow in solving a problem, we are able to look back over the process, refine our methods and store them for use at other times. Skemp (1989) also describes the remediation process as a process of reflecting on the thought processes used in the original work and thus being able to correct it. The writing out of a problem resolution enables this process to take place in problem solving.

2. Beliefs

There is considerable evidence according to Silver (1985) to suggest that one's beliefs affect decision making and determine one's actions.

In the present study, it seems important to investigate the beliefs of students in their pre-service teacher education to ensure that whatever can be done about misconceptions is done before they enter the teaching service. The implied beliefs which became obvious in the process of this study need to be analysed.

Beliefs about mathematics cover three main areas. These are the nature of the subject, its difficulty and its usefulness. The first of these is treated in the N.S.W. Department of Education K - 12 Mathematics Statement of Principles.

This statement clearly claims that mathematics is more than computation, that it is useful and that it is part of our culture. This statement also includes a list of aspects of mathematical learning which would enhance the student's ability to learn mathematics.

The subjects in this study in general appear to agree that mathematics is more than computation, and see it as useful in certain situations. There is evidence to suggest however that even when mathematical thinking is used in the resolution of the problem that this is not perceived as using mathematics.

The difficulty level encountered in learning mathematics covers the full range from easy to hard. The subjects in the current study all found some, though not all, of the problems encountered difficult. The level of difficulty anticipated was to a certain extent determined by the nature of the problem question and the past experience of the subject.

The level of difficulty anticipated by the subjects appears to be linked with the subjects' beliefs concerning their own ability in mathematics. The subjects appear to believe that their ability in mathematics is minimal and that their ability to teach mathematics will be impeded by their lack of mathematical ability. Their approach to teaching the subject will consequently be tentative and less than enthusiastic.

3. Perceptions

"Perception is the process whereby an individual becomes aware of present actuality, and in so doing classifies and structures it" (Skemp, 1979). In the present study, Subject J. linked a belief about mathematics as consisting of numbers and the visual perception of the problem situation. "I hated the question simply because it has numbers because it represents maths per se." In this statement, the subject has become aware of the actuality of numbers, has classified the question under the heading of mathematics, and started to work on it in the way which she thought was expected by relating it to her previous work with similar questions. In this way, this particular subject's perceptions were bound on several occasions to previous experience and proved a barrier to a harmonious resolution to the problem.

4. Language

The references to language, key words and writing ideas down all substantiate the reasonably large research literature on the importance of language in the learning and teaching of mathematics. The suggestion by one of the subjects that the problem solving be a group process also indicates the perceived advantages of discussion in a problem solving situation.

5. Problem Solving Strategies

The knowledge of several different strategies to call upon in the problem solving situation seems to aid the process. Subject J. used seven different strategies to solve the problems and, in all but one, used more than one strategy. On the other hand, Subject P. used only one method, while Subject B. used more than one in each question attempted. All the strategies used were ones which it was reasonable to assume the subjects had developed as a result of previous mathematical experiences.

6. Reasoning Skills

The three subjects were able to display sound reasoning skills, though in the case of Subject P., this was somewhat masked by the level of anxiety exhibited. It was also masked by the subjects' inability to express clearly in a logical sequence the path along which their reasoning had taken them. This provides another example of the importance of good language skills.

CONCLUSION

All the variables considered appear to influence performance in mathematical problem solving. Several of them, for example, beliefs, anxiety, mathematical background and personality characteristics, all impinge on one another. Further investigation is needed to determine the prime influence and isolate strategies to either enhance the positive elements or reduce the effect of the negative ones.

The major areas for more detailed investigation are considered to be the nature and effect of the problem solver's beliefs and the use of the reflective process in the problem solving situation.

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