1NORMAL.STYQUINL90.364 ADDRESSING DIFFICULTIES WITH EARLY ALGEBRA Theory and Practice

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The focus of this study is the difficulties that beginning algebra students experience in understanding the meaning and use of algebraic symbols. This report covers the following aspects:

1.Contrasts derived from viewing algebraic symbols as objects or as variables;

2.Considerations relevant to using certain concrete models for early algebra.

Data COLLECTION

After trialling research instruments and procedures in November 1989, data were collected during 1990 in four schools from 390 students across the secondary grades, Years 7 to 12. Particular attention was given to the sub-group of 208 beginning algebra students in Year 7 by a four-fold strategy: (a) monitoring their first three weeks of classroom work on algebra, (b) administering the test instrument three times during this period, (c) interviewing a selection of the students with regard to their test responses, (d) administering a delayed posttest and conducting some associated interviews. The monitoring of classroom work was organised so that assessment could be made, in terms of theory and practice, of the degree of effectiveness of teaching activities (as in Quinlan et al. 1989) which made systematic use of concrete manipulatives in developing an understanding of the meaning and use of algebraic symbols. Experimental classes were taught by such a concrete approach.

Analyses OF DATA ON ALGEBRAIC SYMBOLS

Scales. Responses to the test were grouped to provide scores on a variety of scale measures. The scale scores were the basis of the analyses which follow.

Views of Letters. Before having any class lessons on algebra the Year 7 students recorded a variety of views of algebraic symbols in the form of letters. The frequency of the following misconceptions rapidly decreased during the testing period for Year 7 students and was also low for students in

Year 9
and above: letters were numbers coded according their position in the alphabet;
letters had no meaning and were to be ignored; letters could be replaced by arbitrary numbers (resulting in the avoidance of algebra by reversion to arithmetic); letters could not be left in an 'answer' (an indication of non-acceptance of lack of closure). Some misconceptions which were more persistent were: letters stand for objects or persons (e.g., 'a' = apple, 'S' = students);
letters were like storehouses for numbers-to-be-discovered (examples of which were simply to be listed). There was gradual growth in the frequency of the more correct views of letters as: representations of numbers of objects (as distinct from the objects themselves); specific unknowns (substitutes for unique numbers, the values of which need not be known); generalized numbers (representing a class of numbers); and numerical variables (members of an abstract mathematical species). The data indicated that the latter view is the most demanding cognitively.

The Objects View Contrasted with the Variables View

Objects or Persons. Letters used as algebraic symbols stand for numbers that can vary, in that form of algebra which is the subject of this study, namely, generalized arithmetic. Regarding letters as standing for objects (e.g., c for a pear) or persons (e.g., P for professors) logically precludes the possibility of their values changing numerically. After three weeks of algebra the control groups were significantly more inclined towards the objects view than were the experimental groups. Variable or Species. Harper's (1979) expectations regarding the acceptance of lack of closure extend those of Collis (1978). Firstly, he says, such acceptance opens the way to the view that the letter has a potential variation across a range of numerical values. Secondly, in order to acquire the concept of a variable, this acceptance needs to extend to regarding the letter as a non-ordered entity which can identify all possible numerals simultaneously and as useful to avoid mentioning all these numerals (p. 242).
This level of thinking is the high point of the continuum of ways for regarding algebraic symbols in early secondary school algebra. Averages on the

Variables Scale for even the top classes reached only 6 out of a possible score of 8. However, the impressive fact was that over 20 percent of the Experimental Group scored 4 or more on this scale after three weeks. Test scores indicated that the experimental groups were developing this concept significantly faster than the control groups. The bulk of the Year 7 students were just beginning to appreciate the concept of letters standing for variables.

Correlations The correlations assembled in Table One are between scale scores for students across Years 7 to 12. The same pattern was obtained using the sub-group scores from the 182 students across Years 8 to 12, and also from the 208 Year 7 students after their first three weeks or so of algebra. A study of these correlations suggests several important conclusions.

Some Preliminary Conclusions from Test Data

Data provided in Table One show that teaching students to regard letters as objects in early algebra may have little influence on their success with substitution exercises and acceptance of lack of closure, despite such misguidance about the true meaning of the letters. There was no statistically significant relationship between the degree to which students accept an objects view of algebraic symbols and their degree of success with substitution and acceptance of lack of closure. However, when they require the notion of variable for success the objects view is likely to be a handicap, as is suggested by the highly significant negative correlation between the level of acceptance of letters as representing objects and the level of development of the variables concept.

Table One Correlations between Scale Scores Years 7 to 12 (N = 390)

Correlations	Objects (J)	Variables (V)
Substitute (S)	0.01 N.S.	0.34 ***
Accept Lack of Closure (ALC)	0.06 N.S.	0.45 ***
Numbers (N)	- 0.76 ***	0.41 ***
Variables (V)	- 0.22 ***	-

N.S. ... Not Significant *** ... Very Significant ($p < 0.001$)

There was a highly significant positive relationship recorded between the level of development of the variables view of algebraic symbols and the level of success with substitution and acceptance of lack of closure, suggesting that while teaching students the variables concept for algebraic symbols teachers would be likely to be also assisting them to succeed in substitution exercises and in accepting lack of closure. Likewise, teaching the view that letters stand for numbers of objects or persons rather than the actual objects or persons is likely to help the development of the concept of variable, as is shown by the highly significant and positive correlation between the scores on the Numbers and Variables Scales.

To gather further analytical information a suitable theory was applied and structured interviews were conducted.

CONCRETE APPROACHES TO ALGEBRA

Structure Mapping. Halford (1987) defines a structure as "a set of elements, with a set of relations or functions defined on the elements. A structure mapping is a rule for assigning elements of one structure to elements of another, in such a way that any functions or relations between elements of the first structure will also be assigned to corresponding functions or relations in the second structure" (p. 611). The analogy "Man is to house as dog is to kennel" is an example of a relational structure mapping. Structure mapping theory seems to be a most suitable tool for analysing the value of any concrete model for helping to develop some concept(s) in mathematics. As Halford and Boulton-Lewis (1989) point out, "the recognition of correspondences between structures ... is central to mathematics learning at all levels" (p.40).

Objects Algebra.

The correlation data discussed above supply empirical evidence for seriously questioning a style of teaching which is widespread in algebra classes, a style which could be called the "objects algebra" style because it uses objects as models for letters in algebra. Teachers using this style use examples such as "What is 2a plus 3a? ... Well it's like adding 2 apples and 3 apples. You get 5 apples, so the answer is 5a." This procedure produces the correct answer and

the mapping from the model to the algebra seems logical, but it is teaching the students that the letter a stands for an apple. The model lacks validity if letters in algebra are to be regarded as representing numerical variables, because an object like an apple cannot represent a variety of possible numbers.
Another common example is "You can't add 2a to 3b because it would be like trying to add 2 apples to 3 bananas." Again, this is teaching students that the letters stand for objects. Short term success by means of using the objects analogy may not assure long term success with algebra. The data summarized in Table One give the warning that using the objects view for letters may seriously impede the development of the notion of the letters as variables.

Brief Description of Models Used by Experimental Classes

Area Model. The unit for the area model was one square centimetre of area. Variables such as 'y' were modelled in terms of the number of square centimetres of area covered by student-selected flat shapes. Models of linear algebraic expressions were constructed on top of centimetre grid paper to emphasize the numerical basis for the model, thus establishing its validity for modelling the form of algebra being taught, namely generalized arithmetic.

Objects-and-containers Model. Similar small objects provided a numerical referent in this model. The 'variables' in a linear expression were identified in terms of the number of objects inside containers, while objects outside containers represented the 'constants'.

Length Model. One centimetre of length was the unit for the length model, allowing variables to be modelled by the number of centimetres of length for selected rods. Comparison of Mappings from Arithmetic and from Concrete Models

An Example. In order to understand the meaning of a functional algebraic expression such as '2y + 5' a student needs to regard 'y' as a variable number and the functional value of '2y + 5' as the value obtained when the value of 'y' is doubled and 5 is added to the result. The structure of the expression is: Double whatever value 'y' has and then add 5. This may look simple enough to experienced mathematicians but the data showed that it is not really simple for all high school students. On the test question "If y = 3, what is the value of 2y + 5" the following were the outcomes: Of 208 Year 7 students after their first three weeks of algebra 85.1 percent correctly wrote '11', the main errors being to write some answer containing 'y' (by 4.8 percent) or to write '28' (by 3.4 percent). Across Years 8 to 12 there were 96.2 percent correct. Let us now compare the mappings required to learn the structure of an expression like this firstly if we start from arithmetic, and secondly if we use one or more of the models described above. Mappings from arithmetic. Mapping from one numerical example to another could be used in an effort to communicate the conventional meaning of '2y + 5' as add 5 to twice the value of 'y'. The idea that 'y' is a numerical variable could result from mapping one or more of these numerical examples onto '2y + 5'. e.g. 2 x 3 + 5 = 6 + 5 could map onto 2 x 4 + 5 = 8 + 5 which could map onto 2 x 0 + 5 = 0 + 5, and each of these could be mapped onto 2 x y + 5 = ... Mapping Using Models. The models used by the experimental classes demonstrate the structure of '2y + 5' with clarity without the necessity to focus on numerical values, even though the numerical aspect of the variable 'y' is validly represented by the models. Using the objects-and-containers model, for instance, the structure of the '2y + 5' is visibly seen as five objects placed beside two containers each holding the same number of objects. Regardless of the number of objects in each of the containers, the structure is always "The total number of objects is 5 more than twice the number in one container". Paralleling the examples above, we have '5 objects near 2 containers each holding 3 objects giving 5 + 6 objects' could map onto '5 objects near 2 containers each holding 4 objects giving 5 + 8', which could map onto '5 objects near 2 containers each holding zero objects giving 5 + 0 objects', and each of these could be mapped onto '2 x y + 5 = ...' A similar visual representation of the structure of the algebraic function would be produced by using the area or length models. A Comparison. While either an arithmetic or model approach may lead to the same level of understanding it was noted that, for experimental and control groups balanced on ability, those who used the models had achieved significantly better on this question about '2y + 5' after the first three weeks of classroom algebra, with 96.7 percent correct as compared with 75.4 percent of the control group.

Some Preliminary Conclusions from Delayed Posttest Interviews

Interviews conducted with one class six months after some of them they had been introduced to algebra with the aid of the area and objects-and-containers model gave the following indications: 1. The students did not think in terms of the models when doing the delayed posttest: Dependence on models was not in evidence; 2. They could still use the models correctly despite no mention or use of them during the previous six months; 3. Students who had gone wrong in the test showed that they could use the models for self-correction; 4. Some were able to generalize from one or more modelled cases to the corresponding algebra. The mappings involved were from models to algebra. In one student interview the function '2n + 8' was modelled in a total of six ways as follows:
with the objects-and-containers model, using n = 6, n = 3, n = 0; with the area model, using n = 6.25; with the length model, using n = 2, n = 3. The student was clearly able to see each of these as an instance of the function '2n + 8';
5. Some who had never used the models before the interview were able to use them with a minimum of introduction, possibly indicating that the processing load involved in understanding the models was not large, although consideration needs to be given here to the fact that these students knew the algebra and were very likely mapping from algebra to models.

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