ERROR PATTERNS IN COMPUTATION

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Introduction

Computational difficulties experienced by students have been studied comprehensively in recent years. In particular, Cox (1975) has identified and classified systematic errors made by students when adding and subtracting whole numbers and Ashlock (1976) has used and extended this approach to suggest methods which may be used to remediate errors. Lankford (1974) has been concerned with analysing students' difficulties by talking to them to determine concepts and methods they possess. In Australia, Newman (1977) has analysed both systematic and non-systematic errors made by sixth-grade students in reading, comprehending, processing and answering a range of mathematical tasks.

There are many possible reasons why some students have difficulty with number work in the primary school. Some students have number work problems which are part of an overall pattern of poor performance at school, often related to low general ability, to low motivation to school generally, or both. Other students have number work problems which are clearly not part of an overall pattern of poor school performance because they are not having problems in other areas of their school work. These latter students would seem to have a specific number work difficulty which may be related to a low numerical aptitude, to the lack of specific knowledges and skills, to lack of motivation specifically for number, to inadequate methods and practices of teaching number in primary school, or to various combinations of these and other possible factors. This study was limited to those students who have problems with using the four operations with whole numbers. It was not concerned with students who experience problems in other areas of mathematics or with students whose number work problems were clearly related to other areas of school work.

One major reason why some students have problems with number work is that they are poor readers and thus have considerable difficulty in reading questions which require them to perform a numerical calculation. Recent work by Newman (1977) indicated that approximately 13 per cent of student errors arose because the Year 6 students with whom she was working were unable to read the questions sufficiently well to understand the meaning of the tasks they were asked to perform. Up to an additional 22 per cent of errors made could have been caused by inadequate comprehension of the questions (Newman, 1977:38). However many students who could read and understand the questions were unable to perform correctly the necessary numerical processes. The largest group of errors, some 26 per cent of the total errors, were made in the application of process skills and these were exclusive of the almost equal proportion of process errors identified as resulting from carelessness or lack of motivation (Newman, 1977:40). Thus the majority of errors, 51 per cent, was related specifically to performance of the numerical process. It was the students who were able to read and to comprehend what they read but who had difficulty with the actual mathematics processes (for whatever reason) who were the concern of this work. A methodology for identifying these students is described in the next section.

The Numeration Test used to survey what students aged 10 years could do (in task-specific terms) has been described in detail elsewhere (ACER, 1976). The major purpose in developing and using the test was to make estimates for Australia as a whole of the proportions of students able to perform a variety of tasks which were seen as important if students were not to be disadvantaged in their everyday lives or in their further progress through the school system. Underlying most of the 33 items in the test was the requirement to perform one or more of the four arithmetic operations correctly with one- and two-digit numerals. Although these and other tests are designed for quite specific purposes they are used frequently in different contexts and for quite different purposes. One question asked in the course of this current work is whether this test, or parts of it, would be useful in identifying individual students having problems in using the four operations. The other major question is whether students having problems with the four operations exhibit an error pattern, or series of patterns, the knowledge of which would be useful in identifying and remediating number work problems. Students were initially identified as having problems if they failed to achieve mastery of the Anchor Items Sub-test. (The eight items of this sub-test are shown in the next section.)

It would be useful if a test such as the one described could provide accountability-type information about student populations as it was designed to do, and also provide information about individuals which could be used to assist students having problems as suggested by Ashlock (1976) who identified a series of common errors made by students in the United States and suggested remedial action for the students concerned.

Selection of Students

Three steps were taken to ensure that the students identified were having difficulties with using the four operations and not with reading or comprehending what was required of them. Firstly the students were given a brief reading comprehension test as well as a test of the four operations. Students who failed to achieve the mastery level set for the Paragraph Reading Sub-test were not included in the group of students studied further. Secondly the number test items used to identify students having difficulties had the minimum of reading associated with them and yet were most basic to any working with numbers. The importance of number facts has been stressed by Ashlock (1976:10). The eight items which constitute the Anchor Items Sub-test have been reproduced below. In the original test form the items were well spaced to allow for students' working.

Add	9 + 6	Answer ______________
Subtract	17 - 9	Answer ______________
Multiply	7 x 6	Answer ______________
Divide	56 ÷ 7	Answer ______________
Add	38 + 19	Answer ______________
Subtract	36 - 19	Answer ______________
Multiply	38 x 9	Answer ______________
Divide	125 ÷ 5	Answer ______________

It can be observed that, if students knew the symbols for the four operations, no reading of words was required by these items. It had been built into the administration of the test that students knew that they should write their answer on the line provided.

Thirdly, students who mastered the Paragraph Reading Sub-test but failed to master the Anchor Items

Sub-test were asked to answer the complete Numeration Test.[1] This was done normally one week after they had answered the eight Sub-test items so provided a useful guide as to the consistency of their responses to these items. Moreover they were asked subsequently to read and explain any other items they could not do correctly to check their ability to read and comprehend the other questions. The other questions had a larger reading component than the Anchor Items questions and two students were dropped from the study at this point because they had reading deficiencies which were contributing significantly to their number work difficulties.

A total of 80 Year 5 students at two schools made up the population of students for this study. At one school which had only 21 Year 5 students, eight students were identified initially as being at least adequate readers but as having problems with number work. A ninth student was also included in this group at the request of the class teacher who believed the student did have problems even though she had mastered the Anchor Items Sub-test. The student was confirmed as having mastered use of the four operations to the level required and was then dropped from this study. In the course of individual tuition two of the original eight students were subsequently found to have reading problems which excluded them also from the group being considered. In one case the student could not read words such as 'length', 'height', 'fourth' and 'balloons' which were required in the numeration test. When the items were read to these two students they could do many of the questions which they had been unable to do previously. How they achieved a mastery level of performance on the reading sub-test in the initial group testing situation is not clear. Perhaps supervision was inadequate. At the other school, nine of the 59 students tested were identified as being adequate readers who had difficulties with number work.

Finally it was determined that a total of 15 students from the 80 considered could read adequately but were having serious problems in applying the four operations accurately. This represented approximately 19 per cent of Year 5 students at the two schools. This proportion of students is comparable with the 15-20 per cent of students aged 10 years with similar problems who were identified in the Australian Studies in School Performance project. These 15 students then received individual tuition in using the four operations. In the course of this work the types of errors the students made were observed and recorded.

The Error Analysis Guidelines

The errors students made were classified according to a modification of an Error Analysis

[1] The mastery level set required that a student answered correctly approximately 80 per cent of all items in each sub-test. As there were four items in the Paragraph Reading Sub-test, three items correct were required for mastery and this level of performance had previously been achieved by only 66 per cent of Australian 10-year-old students (Bourke, 1977a:52). A high level of performance was considered appropriate because of the need to be sure that students were adequate readers before they were selected for case study. The 80 per cent mastery level required that at least six of the eight items in the Numeration Anchor Items Sub-test were required to be correct. This level had been achieved by 70 per cent of 10-year-old students and was considered appropriate because of the basic nature of the number skills involved (Bourke, 1977b:77). For a detailed argument for the mastery concept and the 80 per cent level see Keeves and Bourke (1977:35-41).

Guideline developed by Newman (1977). Newman developed a hierarchy of 'performance strategies' needed to solve written mathematical questions. She then arranged these eight strategies sequentially to form a 'Criterion for Error Causes' instrument to classify student errors. The first two strategies she identified were Reading Ability and Comprehension which included reading and being able to explain the meaning of the words and symbols in the question. These have been excluded in the present study by selecting only students who could read and understand the questions.

Newman's third strategy was Transformation which was concerned with whether the student could select the mathematical operation necessary to answer the question. Again all students in the present study could apply this strategy because of the type of question posed.

The fourth necessary strategy was whether the student could perform the mathematical operation necessary for the task. The process skills involved in using the four operations are at the centre of numerical computation and form the basis of error types being studied here. To Newman's five subdivisions for types of arithmetical error - random response, wrong operation, faulty algorithm, faulty computation and no response - a sixth has been added - inefficient method. An inefficient method is one which, though not incorrect requires the student to become involved in a relatively lengthy, usually iterative process to obtain an answer to one or more steps in a question. The likelihood of error is high. For example a student could (but would be unlikely to) obtain the correct answer to 38 x 9 by counting nine strokes on a page 38 times, and this would be classified as an inefficient method. According to Newman's guidelines, this type of error would be classified as faulty computation. However in her terms a faulty computation which arises from a systematic calculation error (Newman, 1977:35) should result in the same incorrect answer if the student were asked to do the same item again. As the student is likely to obtain a different incorrect answer when he uses an inefficient method, for the present study it was considered desirable to distinguish these errors from errors due to faulty computation. Preliminary work had indicated that there would be many more cases of error of 10-year-old students due to inefficient methods than to faulty computation in the sense used by Newman. The distinction made between efficient and inefficient methods is very similar to that between mature and immature procedures mentioned by Ashlock (1976:1-2).

The student's ability to encode his answer and to write it correctly was the fifth strategy described by Newman. It is possible that a student could give a correct oral answer yet not be able to write the answer down correctly. This type of error should not be confused with the sixth category - that of careless error. Here the error is random and would be unlikely to occur again if the student were given the same task. It is sometimes difficult to distinguish from a process error, especially one due to an intermittently faulty algorithm or to a faulty computation. Some inefficient methods also leave the student prone to error which could be termed as careless in that the same incorrect answer would be unlikely to be repeated. However when a series of different incorrect answers are given to the same question which result from the same type of error (e.g. miscounting) it is more useful to treat the type of error than the particular and variable manifestation of the error.

Motivation was also classified as a separate criterion of error by Newman. This type of error cause is typified by no answer or a random answer when the student subsequently shows that he probably could have answered the questions correctly if he had been motivated to do so. His probable ability to

Table 1 Classification of Errors Made by Fifteen Students Identified as Having Problems with Number Work

Type of Error	Type of Operation				
	Addition	Subtraction	Multiplication	Division	Total (1)
Inefficient method	1	7	10	6	24
Algorithm error	1	7	6	1	15
No response (could not do)	-	3	-	8(2)	11
Careless error	3	1	2	1	7
Faulty computation	-	-	1	-	1
Coding error	1	-	-	-	1
Concept of operation not understood	-	-	-	1	1
Motivation (3)	1	2	2	2	7
Total errors	7	19	21	19	66
No errors	10	2	1	-	13

Note (1) Each student could make more than one type of error.

(2) Some students at the larger school either had not been taught division or were currently being taught this operation. Consequently less probing was done and fewer error types were discovered for division.

(3) These figures result from only two of the 15 students.

made by students at the two different schools. It was clear that attempted use of an inefficient method was by far the major source of error when the four operations are considered together, accounting for more than one third of the total number of errors. This was not so for addition where careless errors made the major contribution to students' problems. In fact there were few errors in addition overall but many more for the other operations. Algorithm errors were also major contributors to problems for subtraction and multiplication. Division was in a special position in that some students had not yet been taught an algorithm for this operation and others were at that time being taught a method for performing division. It could be expected that most of the eight students who could not do the division items at all would have been attempting to use an algorithm for this operation in the following school term.

It is important to note that, almost without exception, the students understood the concepts behind the four operations they were attempting to perform. This suggests that number concepts had been well taught in earlier years at school. On inspection of answers given it seemed on a number of occasions that some students did not understand the concept of place value when carrying in multiplication, whereas they did clearly understand the concept in other numerical contexts. However follow-up questioning and working with these students indicated that they did understand the concept fully but simply failed to implement it consistently in that context. The most likely explanation is that they tended temporarily to be overcome by the task of implementing a new algorithm to the extent that other skills which had been previously learned were eclipsed.

The importance for students of correctly learning one appropriate algorithm and then practising it so eliminate most algorithm errors was evident, especially for subtraction and multiplication.

have done the question when asked is what distinguishes an error resulting from poor motivation from a random response described earlier as a process error.

Finally the form of question should be considered. If the question could reasonably be misinterpreted by a student who otherwise could have done it correctly, then the question is faulty and caused the student's error. Given the nature of the eight questions being considered in this study this possibility was unlikely. In fact none of the students concerned appeared to have misinterpreted any of the eight questions of the Anchor Items Sub-test.

The Participating Schools

Of the two schools participating in the study one was a small school on the fringes of a major Australian State capital city with one class per year level and, in 1978, only 21 students in Year 5. The school served a semi-rural community of small and large farms and some commuters to a light industrial district nearby. Almost all the Year 5 students had been at the school since commencing schooling or shortly thereafter. The Year 5 class teacher was in his second year of teaching, was very popular with his students, was conscientious, cheerful and seemed competent with number. The teacher and other relatively-inexperienced teachers on the staff received encouragement and support from the Principal and a senior teacher who, among other duties, assisted with small group and individual work for students having learning problems. The principal was active in the development of mathematics curricula and methodology for the primary school and so was particularly well placed to assist classroom teachers in this area of work. The Year 5 teacher took his own class for mathematics.

The other was a larger school of some 450-500 students in a disadvantaged area of an outer suburb of the same Australian State capital city. Most parents of students at the school worked in unskilled or semi-skilled occupations, although there was a significant number of skilled tradesmen among parents. Most of the 59 Year 5 students had been at the school for at least two years. Year 5 consisted of two classes - one class teacher being in his second year of teaching and the other in her first year. Both seemed to be conscientious and well-liked by their students and received good support from an interested, active yet realistic Principal. The more experienced teacher seemed competent and confident with number work while the less experienced teacher lacked confidence in teaching some aspects of number, especially when students were having problems. The school had the services of a teacher who was a mathematics consultant in the region and was able to make additional provision for mathematics lessons. The students were divided into three groups - an 'upper' group, a large middle group (taken in turn by the two Year 5 teachers) and a 'lower' group - for part of the time given to mathematics. For the remainder of this time they were in their normal class groups taught by their class teachers. Thus each student received some mathematics instruction designed for his level of performance at that time and some in the normal class where he had an opportunity to advance with the whole group of his peers. One potential disadvantage of an individual student having two or even three different mathematics teachers at the one time will be discussed later in this paper.

Summary of Case Studies

The types of errors made by the 15 students are summarised in Table 1 for the four operations of addition, subtraction, multiplication and division. There were no marked differences in the types of errors

At the larger school where many students had two different teachers for mathematics each week because of the additional provision made for mathematics lessons, for subtraction some students had been taught concurrently the equal additions method and the decomposition method. Neither teacher was aware that this was happening but the result was that two of the fifteen students selected were consistently attempting to use both methods in each subtraction problem. Consequently the students were very confused.

It was also clear that many students resisted using algorithms for some time and returned to iterative methods in which they had more confidence. In general although the teachers concerned thought their students were using the algorithms taught for subtraction and multiplication, many were not. The students considered here were those who were unable to obtain the correct answers - it is highly probable that many who did answer the questions correctly also used inefficient methods but did not come to the attention of the researcher. Teachers who think they are giving students practice in using a newly-taught algorithm by providing appropriate examples and checking answers may often in reality be allowing some students to continue using an inefficient method that will lead to problems later when more difficult computations using the four operations are required.

The Numeration Anchor Items Sub-test did prove to be a useful tool in identifying problems which individual students had with the four operations. In many cases the incorrect answers students gave to items indicated the mistakes in reasoning or computation they had made. However the sub-test is a coarse filter only and needs to be supplemented with additional questions of the same types when any detailed work is attempted with an individual student.

There was consistency or patterns of errors made by most students. When an error was made in computation or in the application of an algorithm, the answer normally indicated the specific error that had been made. An index to possible answers showing likely sources of error for a small number of questions of the types used would be most useful as a rapid method of identifying student errors in the classroom setting. However the inefficient methods described could give almost any answer and thus an index alone would not be sufficient for an error diagnosis to be made. In these cases it was necessary to observe and interact with the student while he was doing the example to identify his problem.

Conclusions

Four related but separate points arise from this work. Those that touch upon consultation and co-ordination between teachers at a school, individualizing of attention given to students and the need to discover patterns of errors within classrooms have, it is suggested, a more general application to curricula than the four operations with whole numbers that have been considered here.

Teachers need to be aware that there are different algorithms which may be applied when using the four operations. Once they are aware of alternative strategies, teachers at any one school should agree on which they will teach. This is advisable because, as students move through the years they have different teachers and, even within a year, some students change classes and/or teachers. Where students are taught mathematics by two or more teachers concurrently such agreement on one strategy is essential. The problem created for students because they had been taught both the equal additions and the decomposition algorithms for subtraction were evident in this study.

Teachers should ensure that students are actually practising the methods they are being taught when given examples for that purpose. The tendency for students to regress to less efficient and ultimately inadequate strategies for using the four operations should not be underestimated. This implies that much closer supervision of what individual students are doing is necessary with follow-up specific to each student's needs. In this study teachers expressed surprise that many of their students were not using the algorithms they had recently been taught. A related point is that perhaps the use of appropriate algorithms for the four operations requires a longer period of supervised practice than most teachers realize.

Ashlock (1976:1) refers to the importance of estimating in helping students avoid errors. Many of the students in this study did make informal estimates of what correct answers should be and consequently were able to recognize obviously incorrect answers. However estimating procedures apparently had not been explained adequately to some students and it is suggested that the importance of teachers encouraging their students to estimate answers be emphasized.

Many teachers may need advice and assistance in recognizing the types of errors students make when looking through the students' working and answers but, in the long run, such investment of effort is worthwhile. Weaknesses in teaching and learning are revealed when errors are seen as results of a pattern with some consistencies and some idiosyncracies rather than as random manifestations of incorrect answers. Further, once a type of error has been identified, the teacher's time can be spent on eliminating the known specific problem, probably concurrently for a number of students who have made errors of that type. However the further development of methods for analysing students' errors systematically will be necessary before simple yet efficient guidelines can be provided for the general use of those concerned with the teaching of the four arithmetic operations.

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