EVALUATION OF THE 'RECONCEPTUALISING EARLY MATHEMATICS LEARNING' PROJECT

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The Pattern and Structure Mathematics Awareness Project (PASMAP) has investigated the development of patterning and early algebraic reasoning among 4 to 8 year olds over a series of related studies. We assert that an awareness of mathematical pattern and structure enables mathematical thinking and simple forms of generalisation from an early age. The project aims to promote a strong foundation for mathematical development by focusing on critical, underlying features of mathematics learning. This paper provides an overview of key aspects of the assessment and intervention, and analyses of the impact of PASMAP on students' representation, abstraction and generalisation of mathematical ideas. A purposive sample of four large primary schools, two in Sydney and two in Brisbane, representing 316 students from diverse socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year and a follow-up assessment in 2010. Two different mathematics programs were implemented: in each school, two Kindergarten teachers implemented the PASMAP and another two implemented their regular program. The study shows that both groups of students made substantial gains on the 'I Can Do Maths' assessment and a Pattern and Structure Assessment (PASA) interview, but highly significant differences were found on the latter with PASMAP students outperforming the regular group on PASA scores. Qualitative analysis of students' responses for structural development showed increased levels for the PASMAP students; those categorised as low ability developed improved structural responses over a relatively short period of time.

Background to the study

Of particular significance in young children's mathematical development are the reasoning processes they use in learning about their world, such as spatial and quantitative reasoning, deduction and induction, analogical reasoning, and statistical reasoning. In essence, effective mathematical reasoning involves the ability to note patterns and structure in both real-world situations and symbolic contexts. Such reasoning enables the formation of generalisations in which the abstraction of ideas and relationships is paramount (Australian Curriculum and Reporting Authority, 2012; National Council of Teachers of Mathematics, 2010). Several related Australian studies have advanced a growing body of research supporting the central role of patterning, structural relationships, and generalisation in early mathematics learning. The Pattern and Structure Mathematics Awareness Project has investigated the development of patterning and early algebraic reasoning over a series of related studies since 2001 (Mulligan & Mitchelmore, in press). The project aims to promote a strong foundation for mathematical development by focusing on critical, underlying features of mathematics learning much earlier than previously thought possible. Other applications include the *Patterns and Early Algebra* (PEAP) *Professional Development* (PD) *Program* (Papic, 2009) and an early numeracy project (Warren & de Vries, 2008) that focus on young children's patterning, early algebraic and

mathematical thinking skills with the aim of closing the gap in numeracy achievement for Indigenous children in rural and regional early childhood settings. This paper reports the findings of a recent evaluation study of a school-entry, year-long mathematics program promoting patterning and structural awareness.

Research on pattern and structure in mathematics learning

Virtually all mathematics is based on pattern and structure. As used here, mathematical pattern means any predictable regularity involving number, space, or measure. Examples include friezes, number sequences, measurement, and geometrical figures. By structure, we mean the way in which the various elements are organised and related. Thus, a frieze might be constructed by iterating a single "unit of repeat"; the structure of a number sequence may be expressed in an algebraic formula; and the structure of a geometrical figure is shown by its various properties. What we may call structural thinking is more than simply recognising elements or properties of a relationship; it involves having a deeper awareness of how those properties are used, explicated or connected (Mason, Stephens, & Watson, 2009).

Structure has been a growing theme in research on children's development of mathematical concepts. Children's understanding of arithmetic has received most attention, with many studies examining counting, grouping, unitising, partitioning, estimating, and notating as essential elements of numerical structure (Hunting, 2003, Lamon, 2002, Wright, 1994). The study of spatial structure has received somewhat less attention. Battista, Clements, Arnoff, Battista, and Borrow (1998) and Outhred and Mitchlemore (2000) have all studied the development of children's understanding of rectangular figures and arrays. A clear relationship to multiplication concepts has been demonstrated. Other structural studies have focused on combinatorial problem solving (English, 1999) and data modelling and statistical reasoning (English, 2012; Lehrer, 2007; Warren & Cooper, 2006).

A recent trend emphasises the possibility of algebraic thinking in early mathematics learning. There is increasing evidence that early algebraic thinking develops from the ability to see and represent patterns and relationships in early childhood. It has also been shown that, given appropriate opportunities, children as young as 4 or 5 years of age can develop pre-algebraic thinking (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006). The Dutch "Curious Minds" project highlights patterning and spatial skills far beyond early numeracy (van Nes & de Lange, 2007).

Early studies on pattern and structure

The study of young children's awareness of mathematical pattern and structure has been pursued since the mid-1990s. Our early studies focused on numerical structures. In our first study (Mulligan & Mitchelmore, 1997), we studied the strategies Grade 2 to 3 children use to solve a wide variety of multiplication and division word problems involving grouping, partitioning, counting and patterning. There followed a study of counting, grouping and place-value knowledge among children in Grades K-6 (Thomas, Mullgan & Goldin, 2002). A further study (Mulligan, Mitchelmore, Outhred & Russell, 1997) focussed on the role of imagery among children in Grades 2-5 as they solved a range of numerical problems involving counting, grouping, base ten structure, and multiplicative and proportional reasoning.

One early study investigated a spatial structure: Children from Grade 1 to Grade 4 were given a variety of partial grids and asked either to complete the grid or to predict the number of squares that would be formed (Outhred & Mitchelmore, 2000). All our later studies have included a wide range of both numerical and spatial structures. In the first study of this nature, we validated a 39-item Pattern and Structure Assessment interview (PASA) in Grade 1 (Mulligan & Mitchelmore, 2009). Various versions of PASA have been used in almost all of our later studies, and a revised version is currently being validated (Mulligan & Mitchelmore, in press).

Subsequent studies have focused on the effects of teaching for pattern and structure on children's structural thinking. In one such study, a research team worked for a year with teachers from Kindergarten to Grade 6 in a New South Wales primary school to scaffold learning with small groups of children within regular classroom time (Mulligan, Mitchelmore, & Prescott, 2006). Another study investigated the effects of a year-long preschool program focused on patterning (Papic, Mulligan & Mitchemore, 2011). A further study explored the effect of a 15-week sequence of pattern-eliciting tasks among a small group of Kindergarten children aged 4 to 6 years who had been identified as having potential difficulties with mathematics learning (Mulligan, Mitchelmore, Marston, Highfield & Kemp, 2008).

A number of related classroom-based studies have emanated from this work. One study investigated children's patterning in three dimensions (McKnight & Mulligan, 2010). Structural development was examined in studies of preschoolers' virtual manipulatives and in a study of programmable robotic toys in terms of young children's representational structure of the dynamic pathways (Highfield, 2010). Goodwin (2009) focused on the stages of structural development in her study of the effect of digital media on young children's concept images of fractions. Each of these studies found that children developed structural understanding in varying ways depending on their attention to pattern and unit iteration, and spatial structuring.

Awareness of Mathematical Pattern and Structure (AMPS)

Background studies have consistently shown that children who show a high structural level on one early mathematics concept tend to show a high level on others. We therefore conjecture that young children possess a general characteristic called *Awareness of Mathematical Pattern and Structure* (AMPS). Children with high AMPS recognise and operate well with a variety of early mathematical patterns and structures, whereas children with low AMPS have difficulty recognising mathematical patterns. We have also found evidence that AMPS is correlated with mathematical achievement. Since mathematics is the study of pattern and structure, it should not be surprising that high AMPS is correlated with high mathematics achievement.

A crucial question is whether AMPS is an inherent characteristic or whether it can be improved through appropriate teaching. Several of our studies indicate that it may be teachable. For example, AUTHOR (2006) found a marked improvement in levels of structural development over a year, particularly in the early grades, as a result of a program with an enhanced emphasis on pattern and structure. In another study (AUTHOR, 2008), AUTHOR showed that nine out of ten Kindergarten children who had been identified as having difficulty in mathematics showed impressive growth in their structural levels after a 15-week program. The clearest evidence for the plasticity of AMPS was obtained in a study by Papic (Papic et al., 2011). Preschool children who received a 6-month intervention program focusing on repeating and spatial patterns outperformed a comparison group across a range of patterning tasks at the end of the intervention. Moreover, their superiority was still clear at the end of their first year of formal schooling, one year later, and even extended to patterns not in the intervention or the subsequent school curriculum.

Method

A purposive sample of four large primary schools, two in Sydney and two in Brisbane, representing 316 students from diverse socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year. At the follow-up assessment in September 2010, 303 students were retained.

Pre and Post Program Intervention Assessment Interviews

All students were administered the *I Can Do Maths* (ICDM) test of general mathematics achievement (Doig & de Lemos, 2000) at the beginning and end of the 2009 school year and again in mid-2010. From pre-test data, two focus groups of five students in each class were selected from the upper and lower quartiles, respectively. These 190 students were interviewed by the research team using a new version of a 20-item Pattern and Structure Assessment (PASA1) in February 2009, a revised 19-item PASA2 in December 2009 (n=184), and the PASA2 and an "extension" PASA in September 2010 (n=170).

The development of the PASA assessment items

The assessment interview sought to complement interview-based numeracy assessment instruments such as the Schedule for Early Number Assessment (NSW DET, 2002) by extending the assessment of counting and arithmetic strategies (addition and subtraction) to multiplicative reasoning. Our earlier studies highlighting the relationship between the development of composite units and unitising, base ten structure, partitioning and multiplicative reasoning influenced the design of the items. Other input included the work of English (1999) on combinatorial thinking and problem solving. Particular attention was paid to representations of 2-dimensional and 3-dimensional arrays and understanding the relationship between unit size and number of units. Patterning tasks were based on our earlier studies with Kindergarten and Year 1 students and Papic's studies with preschoolers. These were extended to include an item integrating multiple counting and emergent functional thinking (Blanton & Kaput, 2005; Warren & Cooper, 2006). The ability to subitise was considered fundamental in developing visual memory and pattern recognition (Hunting, 2003). The inclusion of items on analogical reasoning and transformation was inspired by the work of English (2004), based on the notion that there were strong links between analogical reasoning and spatial patterning. Further, several items required students to draw and explain representations such as the structuring features evident on a clockface. We included another item on drawing a ruler based on our previous analyses of structural development. Additional items such as composite units in 2- dimensional shapes, the structure of ten frames, hundreds charts and counting patterns, the pattern of squares, equivalence and commutativity, and unitising length were formulated for the extension PASA.

The Pattern and Structure Mathematics Awareness Program

Two different mathematics programs were implemented: in each school, two Kindergarten teachers implemented the PASMAP and two implemented their standard program. The PASMAP framework was embedded within but almost entirely replaced the regular Kindergarten mathematics curriculum. A designated researcher/teacher visited each teacher on a weekly basis and equivalent professional development for both pairs of teachers was provided by the research team. A one-day professional development program was provided at the initial stage of the project independently for each teacher group (standard and PASMAP program). The framework was outlined for independent use by the PASMAP teachers, with an accompanying sequence of learning experiences described in terms of syllabus outcomes and core components of the PASMAP. There was sufficient scope in the program for teachers to develop their own teaching/learning sequences that differentiated for individuals. Incremental features of the program were introduced by the research team gradually, at approximately the same pace and with equivalent mentoring for each teacher, over three school terms (May to December). The focus children were monitored closely by the teacher and the research assistant, who collected detailed observation notes, digital recordings of their mathematics learning and work samples, and other classroom-based and school-based assessment data. These data formed the basis of digital profiles for each student.

The program focused on unitising and multiplicative structure, simple and complex repetitions, growing patterns and functions, spatial structuring, congruence, similarity and transformation, the structure of measurement units and data representation. Emphasis was also laid on the development of visual memory and simple generalisation. The program was innovative in its conceptual framework and the way learning experiences were scaffolded, where children were encouraged to seek out and represent

pattern and structure across different concepts and transfer this awareness to other concepts. In other words, the aim was to promote generalisation in early mathematical thinking.

Emphasis was also laid on developing number concepts through pattern and structure such as an emphasis on counting patterns and their relation to measurement, geometry and data exploration, and the structure of mathematical number operations such as equivalence, commutativity and inverse operations. Qualitative analyses of digital recordings and students' representations provided complementary evidence of their invented symbolisations and generalisations particularly in repetitions and growing patterns. Improvements in mathematical processes such as skip counting, multiplicative thinking, unitising and partitioning, similarity and congruence, and area measurement were observed. For example, we tracked the development of individuals' imagistic representations for explicit features of structural development such as unitising, congruence and collinearity.

Data Analyses

Student learning was analysed in three ways:

- Rasch scales were constructed from the various ICDM and the PASA results, based on dichotomous scoring of the ICDM and PASA responses.
- An analysis of covariance was used to compare student performance on the ICDM and PASA assessments in December 2009 and September 2010, again based on dichotomous scoring.
- Students' level of structural development on selected PASA items requiring drawn responses at the three assessment points was analysed.

Other evaluation data includes observations made and artefacts collected during the implementation of PASMAP and teachers' views of the impact of the program on student learning and their own professional learning.

The construction of Rasch scales

The PASA data were analysed to construct a Rasch scale that incorporated graded items along a continuum for students aged 4.5 to 7.5 years. In order to establish the integrity of these items within a single construct, 'Pattern and Structure', it was advantageous to conceptualise these items on a linear scale. The main advantage of using Rasch analysis for constructing the PASA scale was that it could be used to link different versions of the PASA containing different subsets of items. In addition, students' performance on the ICDM, also scored using a Rasch scale, could be later integrated into the one scale to give a broader view of mathematical growth across the three assessment points. Rasch Unidimensional Measurement Models (RUMM) software (Andrich, Lyne, Sheridan & Luo, 2001) was used to generate scale scores for PASA items and student measures for the construction of the PASA scale. Item analysis was used to discard items not functioning well in PASA1 to reformulate PASA2 and the extension PASA. Following this, a separate item map produced for the ICDM scores was integrated into the PASA scale.

The item map indicated that the items and the students were reasonably well matched; in comparison, the ICDM items at the lower end of the scale did not sufficiently challenge the majority of students, although some more difficult ICDM items filled a gap in the scale. The scale's order of item difficulty on PASA items provided a measure of the students' overall level of AMPS. Thus a conceptual analysis of the item and its position on the scale reflected the complexity of the task in terms of pattern and structure as well as the reasoning required to complete it successfully. What we aimed to achieve with the scale was an indicator of AMPS aligned with student ability.

Analysis of covariance

Student scores on the versions of PASA administered at the end of the intervention and at the retention point were analysed using analysis of covariance. The covariates were in each case the initial PASA and ICDM scores, and the factors were school (one of four), ability (high vs low) and program (PASMAP and non-PASMAP.

Analysis of the initial scores PASA and ICDM showed the expected differences between ability levels and the expected equivalence of the two program groups. There was, however, a significant difference between the schools, with the two Brisbane schools scoring lower than the two Sydney schools. No significant interactions were observed.

Analysis of the subsequent ICDM scores indicated no significant interactions or main effects apart from a school effect. In other words, all groups of students in each State made very similar gains on ICDM over the period of the study, but Sydney schools gained more.

Analysis of the subsequent PASA scores gave similar results to ICDM, with one important difference: There was a significant difference between the program groups on each PASA assessment—modest at the end of the intervention ($p < 0.026$), highly significant at the retention point ($p < 0.002$), on each occasion with the PASMAP group scoring higher than the regular group. The difference was, however, only borderline ($p > 0.11$) for the extension PASA. Table 1 provides a summary table of the ANCOVA for the PASA at the retention point.

Table 1

ANCOVA of PASA score at retention point

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1048.432[a]	17	61.672	10.380	.000
Intercept	53.229	1	53.229	8.959	.003
Covariate: PASA	158.346	1	158.346	26.650	.000
Covariate: ICDM	14.071	1	14.071	2.368	.126
School	117.125	3	39.042	6.571	.000
Ability	15.259	1	15.259	2.568	.111
Treatment	61.653	1	61.653	10.376	.002
School * Ability	11.643	3	3.881	.653	.582
School * Treatment	43.663	3	14.554	2.450	.066
Ability * Treatment	.217	1	.217	.037	.849
School * Ability * Treatment	13.589	3	4.530	.762	.517
Error	802.130	135	5.942		
Total	13412.000	153			
Corrected Total	1850.562	152			

a. R Squared = .567 (Adjusted R Squared = .512)

We infer that the PASMAP treatment was effective in promoting the conceptual understanding of early mathematics, as measured by the version of PASA used in this study.

The analysis also shows that NSW students were generally more advanced in their mathematical competencies than the Queensland students. This result is to some extent understandable, since most Queensland students and teachers had not experienced a preparatory curriculum; indeed, 2009 was the first year of a formal mathematics curriculum for 5 year olds. This lack of experience may also explain why the Queensland students also made less progress on both ICDM and PASA over the 18-month period of their formal schooling. Greater progress will probably be made once Queensland teachers become more accustomed to implementing an effective preparatory program.

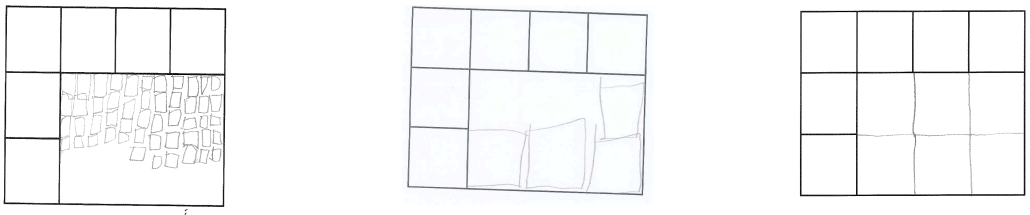
Levels of structural development

The drawn representations made on selected items at the three PASA administrations by 190 students were systematically coded for one of four levels of structural development as follows:

- *Pre-structural*: representations lack evidence of numerical or spatial structure
- *Emergent:* representations show some relevant elements but numerical or spatial structure is not represented
- *Partial structural:* representations show most relevant aspects but are incomplete
- *Structural:* representations correctly integrate numerical and spatial structural features

These levels were developed in previous studies (AUTHOR, 2009), in which coding showed an inter-rater reliability of 0.91.

The analysis indicated marked differences between the program groups in students' levels of structural development (AMPS) at the second and third assessments. Figure 1 shows just one example of how students in the PASMAP group progressed over the 18 months of the study.



March 2009 *Emergent* | November 2009 *Partial structural* | September 2010 *Structural*

Figure 1. A student's responses to grid completion task at the three interview points

Qualitative analysis of the NSW students' profiles and the classroom observation data showed stark differences in the way that the PASMAP students developed their knowledge and reasoning skills. Because the program focused intently on developing structural relationships, only the PASMAP students made direct connections between mathematical ideas and processes, and formed emergent generalisations. For example, students began to link simple multiple counting to more complex multiples, arrays and multiplicative structures through their experience of the notion of unit of repeat in patterning, partitioning, spatial and measurement tasks. Able students used particular features of pattern and structure to build new and more complex ideas. Regular students could also solve tasks requiring multiplicative thinking, but mathematical ideas were often disconnected. For example, they could often not explain what was similar or different between ideas and they rarely formed generalisations.

There was a considerable difference between the gains made within the eight PASMAP classrooms. This result we attribute to variations in the time teachers devoted to the program. Some teachers devoted one 40-minute lesson a week to PASMAP and completed only half of the program. Others taught up to five 1-hour PASMAP lessons a week, revisited structural concepts regularly, and completed almost the entire program. Thus, further analysis of the impact of PASMAP must consider individual teacher when evaluating the program's full impact on students' AMPS.

Conclusions

Our research has established that a large amount of children's mathematical thinking in early childhood can be described in terms of a growing awareness of pattern and structure. Using the Pattern and Structure Pattern and Structure Awareness interview (PASA), we have shown that children's levels of structural development can be reliably categorised, and that each child tends to be at the same level on different tasks typical of the early mathematics curriculum. This finding has led us to formulate the construct of Awareness of Mathematical Pattern and Structure (AMPS) that is prominent in children who achieve highly in mathematics in school and low in those who do not progress easily or develop learning difficulties. We regard the AMPS construct as a significant contribution to research into early childhood education. It provides a lens with which to examine children's thinking at a fundamental level and, in particular, to assess the deeper effects of early mathematics teaching.

The study showed that young students are able to solve a broad range of novel mathematical tasks, including repeating and growing patterns and multiplicative problems, not usually asked of students of this age. Generally, our Kindergarten students were able to construct and use counting and arithmetic strategies up to 20 and beyond, and about 25% of the PASMAP students recognised complex number patterns effectively on a hundreds chart. The ICDM measures could be integrated with the PASA scale to provide a comparative measure, although ICDM assesses numeracy in traditional ways and does little to complement the PASA data.

PASMAP explicitly focuses on the promotion of students' awareness of pattern and structure (AMPS). Particular gains were noted in the related areas of patterning, multiplicative thinking (skip counting and quotition), and rectangular structure (regular covering of circles and rectangles). As expected, a focus on pattern, structure, representation, and emergent generalisation advantaged the PASMAP students. However, students in the regular program were also able to elicit structural responses but had not been given opportunities to describe or explain their emergent generalised thinking that may have been developing. Thus, it was not possible to determine whether more advanced examples of structural development could be attributed to the program or to innate development in the more able students.

Implications

Teaching and learning mathematics through a pattern and structure approach may require fundamental changes to the way that mathematics learning, pedagogy, curriculum and assessment is conceptualised, structured, and implemented. Our ultimate goal is a reliable, coherent model for categorising and describing structural development together with an aligned assessment and pedagogical framework. The PASMAP approach promotes conceptual knowledge that is interrelated and pedagogical strategies that scaffold these interrelationships. Essentially students need to be guided in developing relational thinking by establishing deep connections within and between concepts. This is in contrast to pedagogy that may change attention, sometimes on a daily basis, from one concept to another without opportunity for in depth understanding, and without focusing on the relationships between concepts. However, supporting teachers to implement a structural approach may require that they develop deeper understanding of key mathematical concepts and increased teacher pedagogical content knowledge (PCK). Thus professional learning opportunities would need to be provided and sustained, and the impact on teacher education programs ascertained. The importance of pattern and

structure in mathematics learning is reflected to some extent in the emergence of a national curriculum. However, the key interrelationships between concepts incorporated across the three stands of the *Australian Curriculum–Mathematics* are not foregrounded. However, the Proficiencies (Understanding, Fluency, Problem Solving and Reasoning) support mathematics learning as patterns, relationships and generalisations (ACARA, 2012). A structural approach could support the development of deep conceptual understanding well beyond early algebra, and provide a framework for developing these Proficiencies.

Further Research

A new phase of the research program is currently in progress, *Transforming Children's Mathematical and Scientific Development,* enabling the extension and application of the initial study utilising the same research team. This 3-year longitudinal study integrates the PASMAP pedagogical approach focused on patterns and structural relationships in mathematics to science learning through novel experiences in data modelling and problem solving. An emphasis is placed on developmental features of how students structure data. Students are engaged in an innovative program, usually withdrawn in small groups and taught by the research team in collaboration with the teacher on a weekly basis for a 2-year period. The study tracks three cohorts of students initially employed in the *Reconceptualising Early Mathematics Learning* project when in Kindergarten, through to Grades 2, 3 and 4. Two new cohorts of mathematically able students are being tracked from Kindergarten to Grade 2.

The new project capitalises on young children's potential for developing rich, coherent mathematical and scientific knowledge and documents children's development of core structural ideas and relationships common to mathematical and scientific learning. The development of innovative pedagogical approaches that integrate mathematics and science learning can be realised through systematic longitudinal analyses afforded by the initial project. Our continuing research demonstrates how an interdisciplinary approach (mathematics and science) to the quantitative sciences can be prioritised and integrated within existing practices and curricula at national and international level.

More generally, we aim to validate alternative developmental paths for young children's mathematics and science learning. Further studies will investigate the learning trajectories of students beyond the early years of schooling whose mathematical and scientific reasoning is enhanced by a structural approach. Further our interest lies in the application of the PASMAP approach to assisting those students with special needs; students with low levels of AMPS who may be prone to difficulties in learning mathematics and students with advanced AMPS who are able or gifted at mathematics (AUTHOR, 2011).

We also aim to investigate further the developmental precursors of AMPS. Out studies have consistently found that some children develop powerful mathematical structures and relationships in the prior to school years, while others may be impeded by idiosyncratic imagery throughout their early schooling. Why is this? There are many factors that need investigating, for example, the impact of different early child rearing practices, current instructional practices in early schooling, and possible cognitive-neuroscientific aspects—an emerging field of research in relation to mathematics learning (van Nes & de Lange, 2007).

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