PRE-SERVICE TEACHERS' EXPLANATIONS OF TWO MATHEMATICAL CONCEPTS

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There has been growing interest in the role played in mathematics teaching by teachers' pedagogical content knowledge. This form of teacher knowledge goes beyond mastery of the subject to incorporate how subject matter is actually used in teaching. In particular, it includes knowledge of how to explain concepts and what models can be used to facilitate students' understanding. In this study two items were used to investigate aspects of teachers' mathematical pedagogical content knowledge. Pre-service primary and secondary mathematics teachers were asked to give written reasons to (a) justify the procedure of "adding a zero" onto the end of a whole number when multiplying by 10, and (b) explain the equivalence of 3/8 and 37.5%. Both cohorts of pre-service teachers found the first task difficult, with many struggling to find appropriate language for their explanations. For the second task there was a variety of successful strategies and models, with the pre-service primary teachers having a greater range of pictorial models and the pre-service secondary teachers having more computational strategies. The implications of these results for the preparation of both primary and secondary mathematics teachers will be discussed.

INTRODUCTION

Teaching is a complex activity and the role of teacher knowledge—with its many facets—is known to be central. In mathematics teaching, the role of teachers' pedagogical content knowledge has recently received increased attention. This facet of teacher knowledge is one of five components identified by Shulman (1986) and others, the remaining four being content knowledge, knowledge of students' thinking, curricular knowledge, and knowledge of classroom management. Pedagogical content knowledge incorporates *how* subject matter knowledge is used in teaching, which implies knowing ways of explaining and representing concepts. In mathematics teaching, this includes a knowledge of the representations and models that can be used to depict important mathematical principles (e.g., using multibase arithmetic blocks to illustrate the base ten number system), understanding the connections between fundamental mathematical ideas (e.g., between decimals and fractions), and knowledge of significant examples and counterexamples (e.g., knowing that dividing by a number between 0 and 1 gives an answer bigger than the dividend). The interaction of the five components and the central role played by pedagogical content knowledge has been illuminated recently by Lampert (2001), who, in an extensive case study of her own teaching, depicts the realities of classroom decision-making and shows how a teacher's mathematical knowledge is called upon in the teaching process.

The distinction between knowing mathematics for using it (i.e., content knowledge) and knowing mathematics for teaching it (i.e., pedagogical content knowledge), together with recognition of the need for "Profound Understanding of Fundamental Mathematics", is highlighted in Ma's seminal work *Knowing and teaching elementary mathematics* (Ma, 1999). Her study involved interviews with Chinese and U.S. teachers, in which she explored their abilities to do and explain a variety of elementary mathematics problems. She found that the Chinese elementary teachers, whose education had not continued much beyond high school, generally exhibited deep conceptual knowledge and usually had multiple characterised by a focus on procedures that masked a lack of conceptual understanding, despite the fact that they had studied more advanced mathematics than their Chinese counterparts. This research raises a critical question about what constitutes appropriate preparation for teaching.

It is assumed that pedagogical content knowledge influences teaching and learning, although Ma's study did not examine the teachers' influence on their students' learning. Ma claims, however, that "[t]he real mathematical thinking going on in a classroom, in fact, depends heavily on the teacher's understanding of mathematics" (1999, p. 153). Thompson and Thompson (1996) also argue the importance pedagogical content knowledge, emphasising the need for conceptual *schemes* for teachers, incorporating a clear picture of what materials, activities, and explanations will facilitate the development of mathematical understanding in students. A recent study by Sullivan and McDonough (2002) does provide evidence for the importance of teachers' pedagogical content knowledge in students' learning. They showed that some teachers, whose students made greater gains in learning about length, "seemed able to articulate focused, developmentally appropriate and engaging activities for their students" (p. 255). However, there have been few large-scale studies that have closely examined the nature of the links between teachers' pedagogical content knowledge and what actually takes place in classrooms.

This ability of teachers to articulate not only *what* they are going to do in the classroom, but also the *how* and *why* of mathematical concepts in terms their students will understand is critical. Ball (2000) observes that successful teaching requires an ability to deconstruct mathematical content so as to be able to see it from the learner's perspective. She asserts that "knowing for teaching requires a transcendence of tacit understanding" (p. 245), so that the critical components central to but often invisible in one's own condensed mature knowledge are identified and made explicit. As an illustration, the procedure for simplifying the fraction 10/15 to 2/3 is straightforward, but to explain and justify this "cancelling" process is difficult. Not only is there a challenge to identify these tacit components, but the language required to make this knowledge explicit can be problematic for teachers. This has been illustrated in the work of Thompson and Thompson (1994): the teacher in their study struggled to use language that would convey concepts in terms likely to be understood by students.

These issues provide the background for the current study. In Australia, there has been increased attention on middle school mathematics (Grades 5 to 9), and it seems timely to consider aspects of teachers' pedagogical content knowledge in this area. What makes this particularly critical is that the "middle years" intersect both primary and secondary schooling, and the mathematical backgrounds of teachers in these two sectors vary. Ma's 1999 study suggests that mathematical background may make a difference to teachers' knowledge; does the same apply to primary and secondary teachers whose pre-service experiences differ?

METHOD

Two groups of pre-service teachers undertaking pre-service teacher training in Australia participated in this study. The DipEd cohort (N=29) comprised pre-service secondary mathematics teachers who had studied tertiary-level mathematics to at least sub-major level, prior to undertaking the one-year Diploma of Education in preparation for teaching. The Diploma of Education mathematics method unit that they were undertaking at the time of the study focused on the teaching of secondary level mathematics topics. The BEd cohort (N=38) comprised pre-service primary teachers in the final year of their four-year Bachelor of Education degree. For admission to this degree they had completed mathematics to at least Year 11. Their education course had involved mathematics units that covered elementary mathematics content and pedagogy simultaneously. Most of the mathematical content concerned primary school level mathematics, including fractions, decimals, and place value. When the study was conducted they had completed nearly six semesters of such units. It should be noted that the timing of the study was such that only a third of the available BEd pre-service teachers in the class elected to participate. It is likely that these pre-service teachers would be among the more mathematically confident of that cohort, and so the results for the BEd group may be inflated.

Data for this study came from the pre-service teachers' written responses to two items on a questionnaire investigating pre-service teachers' understanding of middle school mathematics. The whole questionnaire, of 45 minutes' duration, had items on decimals, ratios, fractions, rates, geometry, and probability. Two versions of the questionnaire were made, with most questions common to both. Some items required the solution of a mathematical problem (for pre-service teachers' results on these, see Chick, 2002), whereas the two questions discussed in the present study particularly requested pre-service teachers to *explain* a mathematical concept.

The first of these two questions had place value as its focus, and appeared on both versions of the questionnaire. It asked for an explanation of why "a zero is added" to the end of a whole number when multiplying by 10. Pre-service teachers' responses were categorized according to the reasons used in their explanations, whether the reasons were expressed correctly, and their explanatory power (e.g., some pre-service teachers identified "place value" as a key component but could not clearly express how place value is affected by multiplication by 10).

The second question examined what models and tools pre-service teachers had available for discussing the relationship between fractions and percents. Pre-service teachers were asked to write down three ways of convincing someone that 3/8 is the same as 37.5%. This item appeared on one version of the questionnaire only and so was answered by a sub-sample of the pre-service teachers (n=16 for DipEd; n=20 for BEd). Responses were first categorized by the number of distinctly different explanations a pre-service teacher was able to give, and whether or not the explanations were correct. If two or more of a student's explanations were essentially equivalent they were counted as one response. Individual explanations were then categorized according to the approach used to "explain" the equivalence, and how satisfactory this was as an explanation. Computations utilizing the relationships between fractions and percents (and decimals in some cases) were regarded as correct and satisfactory. Note that some other responses were technically correct—and categorized as such in the first instance—but were unsatisfactory from the point of view of providing a convincing explanation. For example, "cut cake" was regarded as both incorrect and unsatisfactory, because no details were given about what to consider in order to explain the equivalence. In contrast, "cut out 3/8 of a circle and cut out 37.5% of a circle and show they are the same size" was regarded as correct but unsatisfactory because of the impracticality of directly showing 37.5% of a circle without using other relationships.

Neither of the questions specifically suggested that "teaching"-type explanations were being sought. Nevertheless, the wording was intended to provoke explanatory responses and allow pre-service teachers to reveal aspects of their pedagogical content knowledge.

RESULTS

Effect of multiplying by 10

Both cohorts of pre-service teachers had difficulty writing clear explanations for why multiplying by 10 can be carried out by "adding a zero" onto the end of a whole number. The extent of their difficulties can be seen in Table 1. One of the striking outcomes is the similarity in the performance of the two groups. Nearly a quarter of the pre-service teachers did not even attempt an explanation. A further two-thirds gave responses that were unsatisfactory to some extent, with well over 10% only able to give tautological explanations such as “Because it has increased in size by a factor of 10” or “Because if you multiply by 10 it is the same as multiplying by 1 and then putting a 0 on [the] end”.

	DipEd (N=29)	BEd (N=38)
No response	7 (24%)	9 (24%)
Unsatisfactory responses		
Tautological explanation	4	5
Calculation presented but no explanation	0	1
Inadequate “Moving a decimal point”/ “Moving the numbers” explanation	5	5
Inadequate place value explanation	9	11
Other incorrect or unconvincing explanation	4	8
Total students with unsatisfactory responses	18 (62%)	26 (68%)
Satisfactory responses		
Correct “Moving a decimal point”/ “Moving the numbers” explanation	2	1
Correct place value explanation	2	2
Other correct or convincing explanation	1	1
Total students with satisfactory responses	4 (14%)	3 (8%)

Table 1: Pre-service teachers’ explanations for the effect of multiplying by 10 (some responses had both “moving” and “place value” elements and appear in two categories).

Some of the many incorrect or unconvincing explanations are shown below. The first example was accompanied by a comment from the pre-service teacher acknowledging that the explanation was hard to verbalize. The final example makes a valid observation—indeed, pattern recognition in multiples of 10 formed the basis for the two “Other correct or convincing explanations”—but lacks explanatory power with its tautological ending.

> Because number x 10 = 10 x number and number mult[iplied] by 0 is zero. Number mult[iplied] by 1 is number. Therefore [we have] number followed by zero. [DipEd]
>
> Odd or even number when add[ed?] 10 times their unit is always 0. [DipEd]
>
> The number is being increased by 10 units, a lot of ten. [DipEd]
>
> Adding by 10 each time—quicker and easier. [BEd]
>
> Because you are multiplying in a different place value column, therefore there are no “ones” being multiplied (except “1” ten). [BEd]
>
> Multiples of 10 always have a 0 in the units column because they are multiples of 10. [BEd]

Pre-service teachers' past experiences with the mathematical procedure of multiplying numbers, including decimals, by 10 was apparent in some of the explanations. A total of 13 pre-service teachers made some reference to numbers or decimal points "moving" as a result of multiplying by 10. These pre-service teachers talked about "moving the decimal place to the right" or "the digits are 'moved along one space'". What was lacking in many cases, however, was a satisfactory explanation of why this occurs. For example, one BEd pre-service teacher wrote "It is increasing its size by 10, so you move the decimal place to the right to increase the number by 10". Note also the absence of words such as "factor" or "multiple" in talking about "increase". In contrast, a better—but still not entirely satisfactory— explanation was given by a BEd pre-service teacher who wrote:

> When you multiply by ten, you are making the number 10 times bigger. Our place value system is based on multiples of 10, so the added zero shows that the digits have moved up a place. [BEd]

Place value played a role in many explanations (including some of those above that talked about "moving" digits or decimal points, which is why some responses were placed in two categories in Table 1). Again, however, pre-service teachers struggled to give complete explanations.

> To indicate that you are working with tens now and not units. [DipEd]

> The value of the number is being increased by one decimal place. [DipEd]

> "Times by ten" makes all place values increase by one place. [DipEd]

> [Be]cause the place value increases from ones to tens. [BEd]

> Due to place value 4 time[s] 10 is 4 groups of tens. Tens always end in a 0. [BEd]

> Designates that if you are bundling in lots of 10, so if you have 1 lot of 10 you have 10, 2 lots 20, 3 lots 30. [BEd] [Place value teaching often involves "bundling in lots of 10".]

> Because when you multiply by 10 you are gaining another step in the place value. [BEd]

Only about 10% of the pre-service teachers were judged to have given satisfactory responses. The better explanations included the following, although even some of these might be judged to require additional discussion to be correct and complete:

> Because in the position system with base ten, multiplying with 10 moves all the digits one level/position up. Eg, 7 becomes 70, 200 becomes 2000 etc. [DipEd]

> The number represented the number of units[,] now it will represent the number of tens. [DipEd]

> Our system of numbers is "base 10" so multiplying by ten increases the place value by one place. [BEd]

> When you multiply by ten you are making the number 10 times bigger. Our place value system is based on multiples of 10, so the added zero shows that the digits have moved up a place. [BEd]

Equivalence of fraction and percent values

The second question in this study asked pre-service teachers for three explanations for the equivalence of 3/8 and 37.5%. Only a sub-sample of the cohorts was asked and responded to this question. Table 2 shows how many distinct responses pre-service teachers were able to give, and how many were correct.

As can be seen, three pre-service teachers did not give any explanations at all, and there were three DipEd and five BEd pre-service teachers who gave at least one incorrect or unclear explanation (for example, one DipEd pre-service teacher only wrote "Cut cake"). Only nine pre-service teachers, six of whom were DipEd pre-service teachers, were able to give three distinct correct explanations for the equivalence.

	DipEd (n=16)	BEd (n=20)
No response	1 (6%)	2 (10%)
1 explanation — incorrect	1 (6%)	0 (0%)
1 explanation — correct	2 (13%)	4 (20%)
2 explanations — 1 correct/1 incorrect	1 (6%)	1 (5%)
2 explanations — both correct	4 (25%)	6 (30%)
3 explanations — 2 correct/1 incorrect	1 (6%)	4 (20%)
3 explanations — all correct	6 (38%)	3 (15%)

Table 2: Number and correctness of pre-service teachers' distinct explanations for the equivalence of 3/8 and 37.5%.

Table 3 shows the categories of reasons that pre-service teachers gave to explain the equivalence. It includes one explanation which may be soundly based but which had incorrect or unclear elements: the pre-service teacher suggested showing the percentage on a number line from 1 to 100, and 3/8 on a number line from 1 to 10. This response raises a number of queries about the pre-service teacher's understanding, and a follow-up interview would have been useful: there is no discussion about the relative lengths of the number lines, it is not clear why 1 was written instead of 0, it is not clear if 3/8 *of* the number line was meant or 3/8 *on* the number line, and it is not clear why 10 was chosen in preference to 8 or 1, both of which could have been used to good effect.

Some pre-service teachers used correct methods but used results without justification. For example, four pre-service teachers in each of the cohorts merely asserted that 3/8 was equal to 0.375, and some of the pre-service teachers who multiplied 3/8 by 12.5/12.5 did not clearly show the means whereby they obtained 12.5.

The standard algorithm for converting fractions to percents by multiplying the fraction by 100 or 100/1 was utilized by nearly half of the BEd pre-service teachers, which was the predominant computational approach for this cohort. Both groups had a number of other computational strategies, but whereas the DipEd pre-service teachers did not have such a dominant use of the standard algorithm, they used other calculation strategies in greater proportion than the BEd group. For example, the DipEd cohort were more likely to divide 8 into 3 using the division algorithm or to multiply 3/8 by 12.5/12.5 to get 37.5/100. Some computational approaches involved algebraic properties; for example, two pre-service teachers, one from each cohort, multiplied 37.5 by 8/3 to show that the result is 100.

One striking difference between the cohorts is that a larger number of BEd pre-service teachers than DipEd pre-service teachers made use of "pictorial" models, such as an area model that showed both fractions and percents. This usually required the pre-service teacher to equate 1/2 of the area with 50%, and then derive 1/4 and 1/8 as percents. A number of DipEd pre-service teachers did this numerically, without an area model, whereas only one BEd pre-service teacher did so. By contrast, four of the BEd cohort used an area model to derive the same result, using either a circle or a rectangle, compared with one DipEd pre-service teacher.

One particularly effective pictorial model, which came from one of the BEd pre-service teachers, used a 1m strip of paper marked off in cm (thus giving hundredths), which was to be folded into eighths from which 3/8 could be identified. Another BEd pre-service teacher showed two rectangles, one divided into eighths and the other into hundredths, so that 3/8 and 37.5% could be shown and compared. Unfortunately there is some difficulty in directly making this comparison, and the pre-service teacher did not show how this might be resolved.

Type of explanation	Number of responses[1]	
	DipEd (n=16)	BEd (n=20)
Unsatisfactory explanations[2]		
3/8 = 0.375 or 1/8 = 0.125 (correct but without showing how)	4	4
Vague area or equivalence model (some correct but unconvincing)	3	3
Incorrect or unconvincing computation	2	3
“It just is”	0	2
Satisfactory explanations		
Computational use of relationships between fractions and percents (e.g., 3/8 × 100/1)	14	16
Calculated 8 into 3 by long division	4	2
Identified 1/2 as 50%, 1/4 as 25%, and used relationships to get 3/8 as 37.5%	4	1
Area model (circle or rectangle): whole divided into eighths and using 1/2=50%	1	4
Other clear pictorial model	0	2
“Use a calculator to show it”	2	2

Note 1. Some students may have given more than one response of a particular type.
Note 2. Includes the 10 incorrect responses in Table 2 plus some correct responses lacking clarity of explanation

Table 3: Types and number of explanations given for the equivalence of 3/8 and 37.5%.

There were a number of other strategies presented, which, while technically correct, would have practical difficulties in their implementation. Some of these were categorized as “Vague equivalence model”, because it was not clear how these difficulties would be addressed. For example, one pre-service teacher suggested cutting out 37.5% of a circle, and, as a second method, proposed asking word questions such as “If 3 out of 8 like cheese, how many out of 100 would like cheese?” Finally, four pre-service teachers suggested deferring to the expertise of a calculator, which is correct, practical, and may well be convincing (or, at least, as convincing as is the standard algorithm for many people!).

DISCUSSION AND IMPLICATIONS

The results highlight some interesting similarities and differences between the performances of the two cohorts of pre-service teachers, and raise issues concerning the content and approach of courses that prepare them for teaching. After discussing these points, we will conclude by examining the effectiveness and limitations of the research protocol, and consider avenues for future related research.

Performance of the pre-service teachers

The responses of the pre-service teachers to the two questionnaire items were disappointing, particularly with respect to the "multiplying by 10" question. Interestingly, although the DipEd pre-service teachers performed better on the whole questionnaire than the BEd pre-service teachers (see Chick, 2002), Table 1 shows that the performances of the two cohorts on this item were similar. This may reflect the fact that while the BEd pre-service teachers have weaker mathematical backgrounds than the DipEd pre-service teachers they spend time discussing place value in their Bachelor of Education mathematics units.

The pre-service teachers seemed to lack fluency with the language of place value, echoing the concerns of Thompson and Thompson (1994) about the importance of appropriate language. It may be that the fundamental nature of the place value system makes it difficult to explain its properties; moreover, pre-service teachers' long exposure to it may make it so familiar that understanding of it is difficult to deconstruct, illustrating Ball's comment that "expert personal knowledge of subject matter is often ironically inadequate for teaching" (Ball, 2000, p.245).

Judging from some of the responses, many of the pre-service teachers were caught up in being able to "do" some computational procedure—"moving the decimal point", for example—rather than knowing why such a procedure is carried out. It is suspected that from the point of view of some of these pre-service teachers the *procedure* itself is the reason for "adding a zero", yet they may not understand—or at least be able to explain satisfactorily—the underlying mechanism that justifies the procedure. It would be informative to investigate this by interviewing pre-service teachers and probing further.

Although performance on the second question was better, the limited number of models that pre-service teachers had at their disposal was disappointing. One third of the students were unable to come up with even two satisfactory explanations. Some of the responses did, however, reflect some pre-service teachers' well-connected understanding of mathematics. Ma (1999) writes:

> A teacher with [Profound Understanding of Fundamental Mathematics] has a general intention to make connections among mathematical concepts and procedures, from simple and superficial connections between individual pieces of knowledge to complicated and underlying connections among different mathematical operations and subdomains. (p. 122)

As mentioned in the results section, the most noticeable difference between the two cohorts was the greater use of pictorial models by the BEd pre-service teachers in contrast to the emphasis on computational approaches evident within the DipEd group. This difference between the cohorts may reflect the fact that BEd pre-service teachers have more course time available to study alternative representations.

Implications for the preparation of teachers

The results suggest that, in terms of explaining certain fundamental mathematical concepts, there is little difference between pre-service teachers who have a strong mathematical teacher training (the DipEd cohort), and those with fewer years of formal mathematics but a more extensive series of units covering mathematical pedagogy associated with basic concepts (the BEd cohort). What is also apparent, however, is that in both cases this preparation still seems inadequate.

A fundamental component of pedagogical content knowledge seems to be what might be called *conceptual fluency*. This incorporates not only mastery of procedures and the concepts but the ability to articulate them. For many of the US teachers in Ma's (1999) study, conceptual fluency was impeded by a lack of facility with procedures as well as a lack of conceptual understanding. This is often an issue for the BEd pre-service teachers, apparent as they undertake their mathematics education units, although it is not strongly evident in this study. For the DipEd pre-service teachers, procedural ability is not usually an issue, but getting back to the fundamental concepts and being able to articulate them fluently is, as Ball (2000) suggests. This study adds evidence to the idea that mathematics ability is not enough for effective teaching. Clearly time is needed in courses preparing teachers to address fundamental mathematical concepts and explore representations for them in order to develop conceptual fluency. Although the time available for both cohorts is limited we need to examine how teacher education courses can be designed to better help our pre-service teachers develop the necessary pedagogical content knowledge for effective teaching.

Other issues arising from the study

Some comments are necessary about the effectiveness of written items for examining pedagogical content knowledge. Having a written instrument rather than an interview (as used in Ma's 1999 study) offers some practical advantages, including ease of use for large studies, and as an efficient means of gathering at least some preliminary information about teachers' understanding. Nevertheless, there are some shortcomings with requesting written responses, especially as clarification and additional explanation cannot be sought. Furthermore, pre-service teachers may have found it laborious to write down something that they may have more readily expressed verbally. If asked these questions in a real classroom there are many ways in which teachers can provide support for the words of their explanation (e.g., through the use of manipulatives). Furthermore, good teachers can recognize when their students cannot understand their explanations and can give elaboration if required. Some might argue that the use of a time-constrained questionnaire, with no warning about content, is somewhat artificial, with its requirement for spur-of-the-moment responses. It is acknowledged that in planning lessons on a topic teachers have time to anticipate the need for and prepare explanations such as those requested here. Nevertheless, classrooms are unpredictable, and unanticipated questions will arise that teachers with good conceptual understanding *should* be able to answer. Future studies into teachers' pedagogical content knowledge will refine the instrument, and alternative approaches to obtaining data will be tested. One possible strategy is to have a written instrument that teachers can study for a time and respond to, prior to discussing their responses in an interview.

This study's focus on ascertaining aspects of teachers' pedagogical content knowledge is, of course, only half of a very important story. Since we are concerned about the effective teaching and learning of mathematics we must ask, as have many others, how pedagogical content knowledge impacts on the classroom. How does a teacher's conceptual understanding manifest itself in what the teacher does when teaching mathematics, and to what extent and how does it affect pupils' learning outcomes? There has been much research into this, but there are still unanswered questions that warrant further investigation. In particular, there needs to be a closer examination of the links from pedagogical content knowledge to teacher practice and from teacher practice to learning outcomes.

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